

Theory of Functions of a Complex Variable I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says “determine” or “give an example” requires a supporting explanation. Please explain your reasoning in complete sentences.

1. a) Define what it means for a function to be complex-differentiable at a point in \mathbb{C} .
- b) Give an example of a function that is real-differentiable at every point in \mathbb{C} but complex-differentiable at no point in \mathbb{C} .

Solution. A function f defined in a neighborhood of a point z_0 is complex-differentiable at z_0 if and only if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists as a two-dimensional limit. Equivalently, there should exist a function α , continuous at z_0 , such that $f(z) = f(z_0) + \alpha(z)(z - z_0)$. Yet another formulation is that f should be real-differentiable at z_0 , and additionally the real Jacobian matrix should correspond to a complex-linear transformation—in other words, the Cauchy–Riemann equations should hold at z_0 .

One example of a function that is everywhere real-differentiable but nowhere complex-differentiable is \bar{z} , the complex-conjugate function. Being a polynomial in the underlying real coordinates, the function is certainly real-differentiable. That this function is nowhere complex-differentiable can be seen directly from the preceding definition: the limit

$$\lim_{z \rightarrow z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0}$$

does not exist as a two-dimensional limit, for when $z - z_0 \rightarrow 0$ through real values, the fraction has limit 1, while when $z - z_0 \rightarrow 0$ through purely imaginary values, the fraction has limit -1 . An alternate way to see that \bar{z} is nowhere complex-differentiable is to observe that the Cauchy–Riemann equations fail:

$$u(x, y) = \operatorname{Re} \bar{z} = x, \quad \text{and} \quad v(x, y) = \operatorname{Im} \bar{z} = -y, \quad \text{so} \quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

2. a) Prove that if z_1 and z_2 are complex numbers, then

$$|z_1| - |z_2| \leq |z_1 - z_2|.$$

Theory of Functions of a Complex Variable I

b) When does equality hold in this version of the triangle inequality?

Solution. Since $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \operatorname{Re} z_1 \bar{z}_2$, and the real part of a complex number is less than or equal to the modulus of the complex number, it follows that

$$|z_1 - z_2|^2 \geq |z_1|^2 + |z_2|^2 - 2|z_1 \bar{z}_2| = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = (|z_1| - |z_2|)^2.$$

Taking square roots of these nonnegative real quantities shows that

$$|z_1 - z_2| \geq \left| |z_1| - |z_2| \right|, \quad \text{and in particular,} \quad |z_1 - z_2| \geq |z_1| - |z_2|.$$

One necessary condition for equality to hold is that $\operatorname{Re} z_1 \bar{z}_2 = |z_1 \bar{z}_2|$, which means that the quantity $z_1 \bar{z}_2$ is a nonnegative real number. In other words, the points z_1 and z_2 lie on the same ray (half-line) emanating from the origin. A second necessary condition for $|z_1| - |z_2|$ to equal $|z_1 - z_2|$ is that $|z_1| \geq |z_2|$.

Thus equality can hold in the original inequality only if z_1 and z_2 lie on the same ray, and z_2 is no farther from the origin than z_1 is. To put it another way, there exists a real number t such that $0 \leq t \leq 1$ and $z_2 = tz_1$. Evidently this necessary condition is also sufficient for equality: if $z_2 = tz_1$, where $0 \leq t \leq 1$, then $|z_1 - z_2| = (1 - t)|z_1|$, and $|z_1| - |z_2| = (1 - t)|z_1|$.

3. a) Show that if $u(x, y) = 2x - x^3 + 3xy^2$, then u is a harmonic function on the plane \mathbb{R}^2 .
 b) Find an analytic function f such that $\operatorname{Re} f(z) = u(x, y)$. (As usual, $z = x + iy$.)

Solution. (This problem is number 14(ii) on page 10 in Chapter 1.)

Evidently $u_{xx} = -6x$, and $u_{yy} = 6x$, so $u_{xx} + u_{yy} = 0$; thus u is a harmonic function.

You may be able to see by inspection that $u(x, y) = \operatorname{Re}(2z - z^3)$.

If this result does not seem evident to you, then one way to compute f is the following. The Cauchy–Riemann equations imply that $u_x = 2 - 3x^2 + 3y^2 = v_y$ (where $v = \operatorname{Im} f$), so $v(x, y)$ must equal $2y - 3x^2y + y^3 + g(x)$ for some function g . Then $v_x = -6xy + g'(x) = -u_y = -6xy$, which implies that $g'(x) = 0$, so g is a constant function, say c . Accordingly,

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) = 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3 + c) \\ &= 2(x + iy) - (x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3) + ic \\ &= 2(x + iy) - (x + iy)^3 + ic \\ &= 2z - z^3 + ic. \end{aligned}$$

The function f is determined only up to addition of a purely imaginary constant.

Theory of Functions of a Complex Variable I

4. Determine the image of the vertical strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ under the exponential function $\exp(z)$.

Solution. Since $e^z = e^x(\cos y + i \sin y)$, the vertical line on which x takes a constant value c maps to an image circle of radius e^c (which is covered infinitely many times as y varies from $-\infty$ to ∞). As c varies from 0 to 1, these image circles fill up the annulus $\{w \in \mathbb{C} : 1 < |w| < e\}$. This annulus is the image of the indicated strip.

5. Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that the series $\sum_{n=0}^{\infty} |a_n|$ converges, but the series $\sum_{n=0}^{\infty} n|a_n|$ diverges. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is equal to 1.

Solution. (This problem is number 14 on page 17 in Section 2.2.)

Instead of using the formula for the radius of convergence R , it is simpler to reason from the knowledge that if z is a point at which the power series converges, then $|z| \leq R$. When $z = 1$, the power series reduces to $\sum_{n=0}^{\infty} a_n$, which converges by hypothesis (since an absolutely convergent series is convergent). Therefore $1 \leq R$.

Suppose, seeking a contradiction, that the radius of convergence R is strictly greater than 1. Since (by a homework problem) the differentiated series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R as the original power series, this differentiated series would converge absolutely at the point where $z = 1$ (since a power series converges absolutely at points inside the open disk of convergence). That is, the series $\sum_{n=1}^{\infty} n a_n$ would converge absolutely, contrary to hypothesis. The contradiction shows that $R \leq 1$.

Putting together the two inequalities for R shows that $R = 1$.

6. Evaluate the integral $\int_{\gamma} \frac{\sin(z)}{z^4} dz$, where the integration path γ is the unit circle $C(0, 1)$ oriented in the usual counterclockwise direction.

Solution. (This problem is number 3 on page 21 in Section 2.3.)

According to Cauchy's integral formula for derivatives,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-w)^{n+1}} dz$$

when f is analytic on and inside γ and w is a point inside γ . Set the natural number n equal to 3, the point w equal to 0, and the function f equal to the sine function to obtain the following special case:

$$\sin^{(3)}(0) = \frac{3!}{2\pi i} \int_{\gamma} \frac{\sin(z)}{z^4} dz.$$

Theory of Functions of a Complex Variable I

Thus

$$\int_{\gamma} \frac{\sin(z)}{z^4} dz = \frac{2\pi i \sin^{(3)}(0)}{3!} = -\frac{\pi i}{3}.$$

7. Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a simple, closed curve whose trace (that is, image) does not contain the point 0. In less formal language, the point 0 is either inside γ or outside γ but is not on γ . A student proposes the following calculation using integration by parts:

$$\int_{\gamma} \frac{e^z}{z} dz = \int_{\gamma} \frac{1}{z} d(e^z) = \frac{1}{z} e^z \Big|_{\gamma} - \int_{\gamma} e^z d\left(\frac{1}{z}\right) = 0 - \int_{\gamma} e^z \left(-\frac{1}{z^2}\right) dz = \int_{\gamma} \frac{e^z}{z^2} dz.$$

Is the calculation valid? Discuss.

Solution. First of all, observe that the *result* is valid:

$$\int_{\gamma} \frac{e^z}{z} dz = \int_{\gamma} \frac{e^z}{z^2} dz.$$

Indeed, if the point 0 lies outside γ , then both integrals are equal to 0 by Cauchy's theorem. On the other hand, if the point 0 lies inside γ , then Cauchy's integral formula implies that

$$\int_{\gamma} \frac{e^z}{z} dz = 2\pi i e^0 = 2\pi i,$$

while Cauchy's formula for the first derivative implies that

$$\int_{\gamma} \frac{e^z}{z^2} dz = 2\pi i (e^z)'(0) = 2\pi i.$$

The question remains of whether the intermediate steps in the calculation are valid. The issue is whether integration by parts works in the setting of complex path integrals. To see that the answer is affirmative, you can translate the path integral to an ordinary integral over an interval, integrate by parts in that familiar setting, and then translate back to a path integral:

$$\begin{aligned} \int_{\gamma} f(z)g'(z) dz &= \int_a^b f(\gamma(t))g'(\gamma(t))\gamma'(t) dt = \int_a^b f(\gamma(t))\frac{d}{dt}g(\gamma(t)) dt \\ &= f(\gamma(t))g(\gamma(t)) \Big|_a^b - \int_a^b g(\gamma(t))\frac{d}{dt}f(\gamma(t)) dt \\ &= f(\gamma(t))g(\gamma(t)) \Big|_a^b - \int_a^b g(\gamma(t))f'(\gamma(t))\gamma'(t) dt \\ &= f(z)g(z) \Big|_{\gamma} - \int_{\gamma} g(z)f'(z) dz. \end{aligned}$$

When γ is a *closed* curve [$\gamma(b) = \gamma(a)$], the term $f(z)g(z) \Big|_{\gamma}$ evidently vanishes.