Theory of Functions of a Complex Variable I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "determine" or that asks a question requires an explanation to support the answer. Please explain your reasoning in complete sentences.

1. Suppose that a and b are positive real numbers, and $\gamma(t) = a\cos(t) + ib\sin(t)$, where $0 \le t \le 2\pi$. (The path γ is an ellipse.) Show that

$$\int_0^{2\pi} \frac{1}{a^2\cos^2(t) + b^2\sin^2(t)}\,dt = \frac{2\pi}{ab}$$

by evaluating the path integral $\int_\gamma (1/z)\,dz$ in two different ways: by applying Cauchy's integral formula and by using the explicit parametrization of γ.

Solution. (This problem is number 6 on page 12 in Chapter 3.)

According to Cauchy's integral theorem, the path integral $\int_\gamma (1/z)\,dz$ equals $2\pi i$, for the path γ is a simple, closed curve that winds around the origin in the standard counterclockwise direction. On the other hand,

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}\,dt = \int_0^{2\pi} \frac{-a\sin(t) + ib\cos(t)}{a\cos(t) + ib\sin(t)}\,dt$$
$$= \int_0^{2\pi} \frac{(b^2 - a^2)\sin(t)\cos(t) + iab(\sin^2(t) + \cos^2(t))}{a^2\cos^2(t) + b^2\sin^2(t)}\,dt.$$

Since $\sin^2(t) + \cos^2(t) = 1$, taking the imaginary part of the integral shows that

$$2\pi = \int_0^{2\pi} \frac{ab}{a^2\cos^2(t) + b^2\sin^2(t)}\,dt,$$

and the result follows by dividing both sides by ab.

2. If f and g are two entire functions such that $e^{f(z)} = e^{g(z)}$ for all z, then what can you deduce about the relationship between the functions f and g?

Solution. Since $e^{f(z)-g(z)} = 1$, there must be for each point z some integer n_z such that $f(z) - g(z) = 2\pi i n_z$. The analytic function $f(z) - g(z)$ is, in particular, continuous, so the integer n_z depends continuously on z. But the only continuous integer-valued functions are constants. Thus there is some integer n, independent of z, such that the functions $f(z)$ and $g(z)$ differ by $2\pi i n$.

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3. Determine the maximum value of $\operatorname{Re}(z^3)$ when z lies in the unit square $[0, 1] \times [0, 1]$.

   Solution. (This problem is number 23 on page 27 in Chapter 2.)

   In view of the maximum principle for harmonic functions, it suffices to find the maximum of $\operatorname{Re}(z^3)$ when z lies on the boundary of the square. On the bottom edge of the square, the value of z is a real number varying from 0 to 1, so z^3 also varies from 0 to 1, and the maximum of $\operatorname{Re}(z^3)$ equals 1. On the right-hand edge of the square, where $z = 1 + it$ and $0 \le t \le 1$, the real part of z^3 equals $1 - 3t^2$, which varies from 1 to -2; again the maximum is 1. On the top edge of the square, where $z = i + t$ and $0 \le t \le 1$, the real part of z^3 equals $-3t + t^3$, which varies from 0 to -2; the maximum is 0. On the left-hand edge of the square, where z is purely imaginary, the values of z^3 are purely imaginary, so the real part is equal to 0. Consequently, the maximum of $\operatorname{Re}(z^3)$ on the whole square is equal to 1.

4. Prove that if f is an entire function such that $|f(z)| > 1$ for all z, then f must be a constant function.

   Solution. (This problem is number 13 on page 27 in Chapter 2.)

   Evidently f never takes the value 0, so $1/f$ is an entire function. Moreover, the hypothesis implies that $1/f$ has modulus bounded above by 1. By Liouville's theorem, the bounded entire function $1/f$ must be constant, so f must be constant too.

5. Suppose f is analytic in the disk where $|z| < 10$, the modulus $|f(z)|$ is bounded above by 28 for all z in this disk, and $f(0) = 0$. How large can $|f(5)|$ be?

   Solution. Define a function g as follows:

   $$g(z) = \frac{f(10z)}{28} \qquad \text{when } |z| < 1.$$

   Then g is analytic in the unit disk, $g(0) = 0$, and $|g(z)| \le 1$ when $|z| < 1$. By the Schwarz lemma, $|g(z)| \le |z|$ when $|z| < 1$. Consequently,

   $$|f(5)| = |28g(1/2)| \le 28/2 = 14.$$

   The upper bound 14 cannot be improved, for this bound is attained when $f(z) = 14z/5$.

6. Show that there does *not* exist an analytic function f in the disk $D(0, 2)$ such that

   $$f(1/n) = (-1)^n/n \qquad \text{for every natural number } n.$$

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Solution. Seeking a contradiction, suppose that such a function f does exist. Taking n to be even shows that the function $f(z)$ and the function z take the same value at the point $1/(2k)$ when k is a natural number. This sequence of points accumulates at the center of the disk, so by the identity theorem, the function $f(z)$ must be identically equal to the function z. On the other hand, the function $f(z)$ matches the function $-z$ at the point $1/(2k+1)$ for each natural number k, so the identity theorem implies that $f(z)$ must be identically equal to $-z$. Contradiction.

7. Suppose f is a nonconstant analytic function in the unit disk $D(0,1)$, not necessarily one-to-one, and the image of f is a region Ω. Must Ω be a simply connected region?

Solution. If f is not one-to-one, then the image region Ω need not be simply connected. For example, suppose $f(z) = \exp(8iz)$. Then the restriction of f to the interval $[0, \pi/4]$ of the real axis defines a simple, closed curve whose trace (image) is the unit circle $C(0,1)$. But 0 is a point outside Ω (since the exponential function never takes the value 0). Thus the region Ω contains a closed path that has nonzero winding number around a certain point outside Ω, so Ω is not simply connected.