Theory of Functions of a Complex Variable I

Instructions Please solve the following six problems (all of which are exercises excerpted from the textbook). Treat these problems as essay questions: supporting explanation is required.

1. Find every complex number z satisfying the property that $|z - i| = 2z + i$.

2. Let ω denote $e^{2\pi i/3}$, a cube root of 1. Let $g(z)$ denote the product
$$\cos(z)\cos(\omega z)\cos(\omega^2 z).$$
Show that when n is not a multiple of 3, the nth coefficient in the Maclaurin series of g is equal to zero. In other words, there exists an entire function f such that $g(z) = f(z^3)$ for every z.

3. Let C be the circle with center 1 and radius 1 (oriented in the usual counterclockwise direction and traversed once). Evaluate the line integral
$$\int_C \frac{1+z}{1-z}\,\mathrm{d}z.$$

4. Suppose f is an entire function such that
$$\int_0^{2\pi} |f(re^{i\theta})|\,\mathrm{d}\theta$$
is a bounded function of the radius r. Show that f must be a constant function. (This problem is a variation on Liouville's theorem.)

5. Use the residue theorem to prove that
$$\int_0^\infty \frac{1}{x^4 + 6x^2 + 8}\,\mathrm{d}x = \frac{\pi(\sqrt{2} - 1)}{8}.$$

6. Determine the number of zeroes of the function $2iz^2 + \sin(z)$ in the rectangle where $|\operatorname{Re}(z)| \le \pi/2$ and $|\operatorname{Im} z| \le 1$.
(You may find it useful to know that $\cosh(1) < 1.6$.)

Optional bonus problem for extra credit

Let D denote $\mathbb{C} \setminus [-1, 1]$, the complex plane with the closed segment $[-1, 1]$ of the real axis removed. Is it possible to define a holomorphic branch of $\sqrt{z^2 - 1}$ on D? In other words, does there exist a holomorphic function f on D with the property that $(f(z))^2 = z^2 - 1$ for every z in D? Explain why or why not.