Theory of Functions of a Complex Variable I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "determine" or "give an example" requires a supporting explanation. Please explain your reasoning in complete sentences.

1. a) Define what it means for a function to be complex-differentiable at a point in $\mathbb{C}$.
   b) Give an example of a function that is real-differentiable at every point in $\mathbb{C}$ but complex-differentiable at no point in $\mathbb{C}$.

2. a) Prove that if z_1 and z_2 are complex numbers, then
   $$|z_1| - |z_2| \leq |z_1 - z_2|.$$
   b) When does equality hold in this version of the triangle inequality?

3. a) Show that if $u(x, y) = 2x - x^3 + 3xy^2$, then u is a harmonic function on the plane $\mathbb{R}^2$.
   b) Find an analytic function f such that $\operatorname{Re} f(z) = u(x, y)$. (As usual, $z = x + iy$.)

4. Determine the image of the vertical strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \}$ under the exponential function $\exp(z)$.

5. Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that the series $\sum_{n=0}^{\infty} |a_n|$ converges, but the series $\sum_{n=0}^{\infty} n|a_n|$ diverges. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is equal to 1.

6. Evaluate the integral $\displaystyle\int_\gamma \frac{\sin(z)}{z^4}\,dz$, where the integration path γ is the unit circle $C(0, 1)$ oriented in the usual counterclockwise direction.

7. Let $\gamma\colon [a, b] \to \mathbb{C}$ be a simple, closed curve whose trace (that is, image) does not contain the point 0. In less formal language, the point 0 is either inside γ or outside γ but is not on γ. A student proposes the following calculation using integration by parts:
   $$\int_\gamma \frac{e^z}{z}\,dz = \int_\gamma \frac{1}{z}\,d(e^z) = \frac{1}{z}e^z \Big|_\gamma - \int_\gamma e^z\,d\left(\frac{1}{z}\right) = 0 - \int_\gamma e^z \left(-\frac{1}{z^2}\right) dz = \int_\gamma \frac{e^z}{z^2}\,dz.$$
   Is the calculation valid? Discuss.