Final Examination

Instructions: Write solutions to any **six** of the following seven problems. If you attempt all seven problems, please indicate which six you want graded.

1. State
   (a) Cauchy's formula for the radius of convergence of a power series,
   (b) some version of Cauchy's integral formula, and
   (c) some theorem (from this course) *not* named after Cauchy.

2. Consider the following complex numbers:
   (a) $(1+i)^{2018}$ (b) $\left(\sqrt{3}+i\right)^{12}$ (c) the principal value of 12^i.
   Which of these three complex numbers has the largest real part? Explain how you know.

3. Suppose $\gamma : [0,1] \to \mathbb{C}$ is the unit circle, that is, $\gamma(t) = e^{2\pi i t}$. Give an example of an analytic function f on $\mathbb{C} \setminus \{0\}$ such that
$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = 12 \qquad \text{and} \qquad \frac{1}{2\pi i}\int_\gamma (f(z))^2\,dz = 2018.$$

4. Determine the value of $\sup\left\{ \left|\frac{\sin(z)}{z}\right| : \quad z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1 \right\}$.

5. According to the "method of partial fractions," there exist constants $A_1, A_2, \ldots, A_{2018}$ such that
$$\frac{z^{12}}{(z-1)(z-2)\cdots(z-2018)} = \frac{A_1}{z-1} + \frac{A_2}{z-2} + \cdots + \frac{A_{2018}}{z-2018}.$$
   The coefficient A_n is simply the residue of the left-hand side at the simple pole n. Prove that $A_1 + A_2 + \cdots + A_{2018} = 0$.

6. Suppose f is an entire function having the property that $f(z^{12}) = (f(z))^{2018}$ for every z. Prove that f must be a constant function.

7. A student analyzes the real integral $\int_0^\infty e^{-x^4}\,dx$ as follows. "Make the change of variable $x = it$. Then $x^4 = t^4$ and $dx = i\,dt$. When x goes from 0 to ∞, so does t, whence
$$\int_0^\infty e^{-x^4}\,dx = i\int_0^\infty e^{-t^4}\,dt.$$
   The name of the dummy integration variable does not matter, so $\int_0^\infty e^{-x^4}\,dx$ equals i times itself, hence equals 0. But WolframAlpha says that $\int_0^\infty e^{-x^4}\,dx = \Gamma(5/4) \approx 0.906$. Did I just show that mathematics is inconsistent?" Restore the student's sanity for the holidays by explaining the flaw in the reasoning.