Final Examination

Instructions Solve *six* of the following eight problems (most of which are taken from the textbook).

1. State the following theorems (with precise hypotheses and conclusions): Riemann's theorem on removable singularities, the homology version of Cauchy's theorem, and one of Picard's theorems.

2. Find both the real part and the imaginary part of $\left(\dfrac{-1-i\sqrt{3}}{2}\right)^6$.

3. Let G be the complex plane slit along the negative part of the real axis: namely, $G = \mathbb{C} \setminus \{ z \in \mathbb{R} : z \le 0 \}$. Let n be a positive integer. There exists an analytic function $f : G \to \mathbb{C}$ such that $z = (f(z))^n$ for every z in G. Find every possible such function f.

4. Evaluate $\displaystyle\int_\gamma \frac{e^{iz}}{z^2}\,dz$ when γ is the unit circle traversed once counterclockwise [that is, $\gamma(t) = e^{it}$ and $0 \le t \le 2\pi$].

5. Let r be a positive real number, let θ be a real number, and let γ be a continuously differentiable curve in $\mathbb{C} \setminus \{0\}$ that begins at the point 1 and ends at the point $re^{i\theta}$. Prove the existence of an integer k such that

$$\int_\gamma \frac{1}{z}\,dz = \ln(r) + i\theta + 2\pi i k.$$

6. Suppose $f(z) = \dfrac{1}{1-e^z}$. Find the singular part of f at each singular point.

7. Let G be a bounded open set. Suppose the function f is continuous on the closure of G and analytic on G. Additionally, suppose there is a positive constant c such that $|f(z)| = c$ for every z on the boundary of G. Prove that either f is a constant function, or f has at least one zero in G.

8. Apply Rouché's theorem to prove the fundamental theorem of algebra (every polynomial of degree n has n zeroes, counting multiplicity).