Recap: The complex derivative

Suppose G is an *open* subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a function, and p is a point of the set G. The statement "f is (complex) differentiable at p" means any one of the following equivalent properties.

1. $\displaystyle\lim_{z\to p} \frac{f(z) - f(p)}{z - p}$ exists (as a complex number, not ∞).
2. There exists a complex-linear function $\ell\colon \mathbb{C} \to \mathbb{C}$ such that
$$\lim_{z\to p} \frac{f(z) - f(p) - \ell(z - p)}{z - p} = 0.$$
3. There exists a function $\widetilde{f}\colon G \to \mathbb{C}$, continuous at p, such that $f(z) - f(p) = \widetilde{f}(z)(z - p)$.

The *derivative* $f'(p)$ means the value of the limit in property 1, and $\ell(z)/z$ in property 2, and $\widetilde{f}(p)$ in property 3.

Examples

1. If $G = \mathbb{C}$, and $n \in \mathbb{N}$, and $f(z) = z^n$, then f is differentiable at every point, and $f'(p) = np^{n-1}$ for every p. Indeed

$$\begin{aligned}\lim_{z \to p} \frac{f(z) - f(p)}{z - p} &= \lim_{z \to p} \frac{z^n - p^n}{z - p} \\ &= \lim_{z \to p} (z^{n-1} + z^{n-2}p + \cdots + zp^{n-2} + p^{n-1}) = np^{n-1}.\end{aligned}$$

2. If $f(z) = \overline{z}$ (complex conjugate), then f is (complex) differentiable at *no* point. Indeed,

$$\lim_{z \to p} \frac{f(z) - f(p)}{z - p} = \lim_{z \to p} \frac{\overline{z} - \overline{p}}{z - p}.$$

When $z \to p$ horizontally, the limit equals 1, but when $z \to p$ vertically, the limit equals -1. So the limit does not exist.

Terminology

Differentiability is a property that takes place at a point.

A function that is (complex) differentiable at every point of an open set G is called *analytic* on G or *holomorphic* on G. (Cauchy's own terminology of "synectic functions" and "monogenic functions" is obsolete.)

When a set S is not open, "analytic on S" means "analytic on some (unspecified) open set containing S."

Reminder on real differentiability in $\mathbb{R}^2$

Express $f(z)$ as $u(x,y) + iv(x,y)$, where $z = x + yi$.

Viewed in this way as a function on $\mathbb{R}^2$, the function f is differentiable in the *real* sense at a point (a,b) if there exists a linear transformation T of $\mathbb{R}^2$ such that

$$\lim_{(x,y)\to(a,b)} \frac{\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - \begin{pmatrix} u(a,b) \\ v(a,b) \end{pmatrix} - T\begin{pmatrix} x-a \\ y-b \end{pmatrix}}{|x-a| + |y-b|} = 0.$$

The transformation T is represented by the Jacobian matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \qquad \text{evaluated at the point } (a,b).$$

Real differentiability versus complex differentiability

If $f(z) = u(x, y) + iv(x, y)$, and f is real differentiable, then f is complex differentiable iff the real Jacobian matrix corresponds to a complex-linear transformation.

By a homework exercise, this property means that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(note the sign)},$$

the so-called *Cauchy–Riemann equations*.

Some theorems about the Cauchy–Riemann equations

Suppose G is an open subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a function, and $f(z) = u(x,y) + iv(x,y)$, where $z = x + yi$.

1. If u and v have continuous first-order partial derivatives, then f is analytic on G iff the Cauchy–Riemann equations hold on G. [Theorem 2.29 in Chapter III]
2. If f is analytic, then u and v do have continuous partial derivatives [see Corollary 2.12 in Chapter IV], and the Cauchy–Riemann equations hold.
3.
4.

Assignment due next class

1. Suppose G is the punctured plane $\mathbb{C} \setminus \{0\}$, and $f(z) = 1/z$. Show in two ways that f is analytic on G:
   (a) Apply one of the definitions of the derivative to show that $f'(z)$ exists and equals $-1/z^2$ when $z \neq 0$.
   (b) Express $f(z)$ in the form $u(x, y) + iv(x, y)$ and verify the Cauchy–Riemann equations.
2. Show that if $f(z) = |z|^2$, then f is analytic on no open subset of $\mathbb{C}$. (This problem is a reinterpretation of Exercise 1 in §2 of Chapter III.)