Methods for constructing analytic functions

- Infinite series. Example: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- Infinite products. Example: $\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$
- Integrals. Example: $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$, $\mathrm{Re}(z) > 0$ (Gamma function)
- Differential equations. Example: $J_1(z)$ is a solution of the equation $z^2 \frac{d^2 J_1}{dz^2} + z \frac{dJ_1}{dz} + (z^2 - 1) J_1 = 0$ (Bessel function of order 1)

Existence of the radius of convergence

Theorem

There are three possibilities for a power series $\sum_{n=0}^{\infty} c_n z^n$.

1. *The series converges only when* $z = 0$.
2. *The series converges for every complex number* z.
3. *There is a positive real number* R *such that the series converges when* $|z| < R$ *and diverges when* $|z| > R$.

Proof.

If $w \neq 0$, and $\sum_{n=0}^{\infty} c_n w^n$ converges, then the numbers $|c_n w^n|$ must be bounded. Now $|c_n z^n| = |c_n w^n|\,|z/w|^n \leq (\text{constant})|z/w|^n$, so if $|z| < |w|$, then $\sum_{n=0}^{\infty} c_n z^n$ converges (absolutely) by comparison with the geometric series $\sum_{n=0}^{\infty} |z/w|^n$. So $\sum_{n=0}^{\infty} c_n z^n$ converges in a union of disks centered at the origin. □

Cauchy's root test (1821)

Théorème I. — *Cherchez la limite ou les limites vers lesquelles converge, tandis que n croît indéfiniment, l'expression $(u_n)^{\frac{1}{n}}$, et désignez par k la plus grande de ces limites, ou, en d'autres termes, la limite des plus grandes valeurs de l'expression dont il s'agit. La série (1) sera convergente si l'on a $k < 1$, et divergente si l'on a $k > 1$.*

Theorem (English interpretation)

Suppose $u_n > 0$ for every n, and let k denote $\limsup_{n\to\infty} u_n^{1/n}$. The series $\sum_n u_n$ converges if $k < 1$ and diverges if $k > 1$.

Cauchy's formula for the radius of convergence

Let A denote $\limsup_{n\to\infty} |c_n|^{1/n}$. The radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$ equals

$$\begin{cases} \infty, & \text{if } A = 0, \\ 0, & \text{if } A = \infty, \\ \dfrac{1}{A}, & \text{otherwise.} \end{cases}$$

Proof.

Apply the root test for convergence of series, observing that $\limsup_{n\to\infty} |c_n z^n|^{1/n} = |z| \limsup_{n\to\infty} |c_n|^{1/n}$. □

Assignment due next class

1. Read §1 and §2 of Chapter III.
2. Solve parts (a), (b), and (d) of Exercise 6 in §1 of Chapter III (page 33).