Warm-up exercise

1. What are all the possible values of $\log(1+i)$ for different branches of the logarithm?
2. What are all the possible values of i^i, that is, $e^{i\log(i)}$?

Terminology: Paths

- A *path* in a region G means a continuous function whose domain is a closed, bounded interval (in $\mathbb{R}$) and whose image is a subset of G.
  Example: If $\gamma(t) = e^{2\pi i t}$, then $\gamma\colon [0,1] \to \mathbb{C}$ is a path, "the unit circle traversed counterclockwise."
- A path γ is *smooth* if the (real) derivative γ' is continuous (γ' is a one-sided derivative at an endpoint of the interval).
  Example: If $\gamma(t) = t^{3/2} + i\sin(t)$, then $\gamma\colon [0,1] \to \mathbb{C}$ is a smooth path.
- A path γ is *piecewise* smooth if the domain interval can be partitioned into finitely many subintervals on each of which γ is smooth.
  Example: If $\gamma(t) = t + i|t|$, then $\gamma\colon [-1,1] \to \mathbb{C}$ is a piecewise smooth path.

A subtlety

Consider two versions of the unit circle:
$\gamma_1 \colon [0,1] \to \mathbb{C}$, $\gamma_1(t) = e^{2\pi i t}$, and
$\gamma_2 \colon [0,2\pi] \to \mathbb{C}$, $\gamma_2(t) = e^{it}$.

These are different paths (different functions) representing the same geometric object in $\mathbb{C}$.

A *curve* is an equivalence class of paths, the equivalence relation being reparametrization.

In practice, the distinction between paths and curves is usually ignored.

Rectifiable paths

A path γ is *rectifiable* if the path has finite length in the following sense.

A *partition* of the domain interval of γ is a finite number of points t_0, t_1, ..., t_n such that $t_0 < t_1 < \cdots < t_n$, and t_0 is the left-hand endpoint, and t_n is the right-hand endpoint.

If the sum $\sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|$ has a finite upper bound, independent of the partition, then γ has *bounded variation*.

The least upper bound of these sums is the *total variation* of γ, defined to be the length of the path.

An example and a non-example

- From real calculus, you know the length of a smooth parametric curve given by $x(t)$ and $y(t)$, $0 \leq t \leq 1$: namely,

$$\int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

  In the language of complex analysis, this formula says that a smooth path $\gamma\colon [0,1] \to \mathbb{C}$ is rectifiable and has length $\int_0^1 |\gamma'(t)|\, dt$.   [Proposition 1.3 in Chapter IV]
- Giuseppe Peano (1858–1932) constructed the first "space-filling curve," a continuous non-rectifiable path whose image is a square.
  [Sur une courbe, qui remplit toute une aire plane, *Mathematische Annalen* **36** (1890), no. 1, 157–160]

Assignment due next class

- Read the beginning of §3 of Chapter III, through page 47.
- Solve Exercise 5 on page 54, which asks for the fixed points of dilation, translation, and inversion on the extended complex numbers.
- Solve Exercise 1 on page 67 about bounded variation.