Reminder

The first examination takes place in class on Thursday, October 4.

Please bring paper and a writing implement to the exam.

Integration along a path

If $\gamma\colon [a, b] \to \mathbb{C}$ is a smooth path, and f is a continuous function on the image of γ, then $\int_\gamma f(z)\,dz$ means $\int_a^b f(\gamma(t))\gamma'(t)\,dt$.

More generally, if the path γ is not smooth but has bounded variation, then $\int_\gamma f(z)\,dz$, or $\int_\gamma f$, or $\int_a^b (f \circ \gamma)\,d\gamma$ means the limit of *Riemann–Stieltjes sums*

$$\sum_{k=1}^{n} f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})),$$

where the points $t_0, \ldots, t_n$ form a partition of $[a, b]$, and τ_k is an arbitrary point between t_{k-1} and t_k.

Remark: For most purposes, piecewise smooth paths suffice, for Lemma III.1.19 says that every rectifiable path can be arbitrarily well approximated by a polygonal path.

The mean-value property of analytic functions

Suppose f is analytic on a disk $\{\, z \in \mathbb{C} : |z| < R \,\}$. Consider

$$\frac{d}{dr}\left(\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta\right), \qquad \text{where } 0 < r < R.$$

Assuming that f has a continuous derivative, apply Leibniz's rule for differentiating an integral [Proposition IV.2.1] to rewrite as

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f}{\partial r}\,d\theta, \qquad \text{or} \qquad -\frac{i}{2\pi r}\int_0^{2\pi} \frac{\partial f}{\partial \theta}\,d\theta$$

by the Cauchy–Riemann equations, hence 0.

Therefore $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(re^{i\theta})\,d\theta$ is a constant function of r, hence equal to the limit as $r \to 0$, which equals $f(0)$.

Application 1: Cauchy's integral theorem on a disk

Apply the mean-value property to the function $z\,f(z)$:

$$0 = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, re^{i\theta}\, d\theta = \frac{1}{2\pi i}\int_0^{2\pi} f(re^{i\theta})\,\frac{\partial}{\partial\theta}\big(re^{i\theta}\big)\, d\theta.$$

Consider the circular path $\gamma_r\colon [0,2\pi]\to\mathbb{C}$, $\gamma_r(\theta) = re^{i\theta}$.

The integrand above is $f(\gamma_r(\theta))\,\gamma_r'(\theta)\,d\theta$.

So $\displaystyle\int_{\gamma_r} f(z)\,dz = 0$ for every function f that is analytic in a disk of radius larger than r.

Application 2: Special case of Cauchy's integral formula

Rewrite the mean-value property as a path integral over γ_r:

$$\begin{aligned}
f(0) &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi i}\int_0^{2\pi} f(\gamma_r(\theta))\frac{\gamma_r'(\theta)}{\gamma_r(\theta)}\,d\theta \\
&= \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z}\,dz.
\end{aligned}$$

An important Möbius transformation

Let D be the open unit disk, $\{ z \in \mathbb{C} : |z| < 1 \}$. When $a \in D$, let $\varphi_a(z)$ be $\dfrac{a - z}{1 - \bar{a}\,z}$. Then φ_a is self-inverse, maps D bijectively to D, and maps ∂D bijectively to ∂D.

Proof.

A computation, the main step being

$$\left|\frac{a - z}{1 - \bar{a}\,z}\right|^2 = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}\,z|^2}.$$

The self-inverse property follows because $\varphi_a \circ \varphi_a$ fixes the four points 0, a, $1/\bar{a}$, and ∞. A non-identity Möbius transformation can fix at most two points in the extended complex numbers. □

A tricky application of φ_a

Apply the special case of Cauchy's integral formula to the composite function $f \circ \varphi_a$ when $r = 1$:

$$f(a) = f \circ \varphi_a(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f \circ \varphi_a(z)}{z} \, dz.$$

Trick 1: Reparametrize the path: $z = \varphi_a(w)$.

Trick 2: Compute $\dfrac{dz}{z}$ *locally* as $d \log(z)$, or $d \log \varphi_a(w)$, or

$$\left(\frac{1}{w - a} + \frac{\bar{a}}{1 - \bar{a}\, w} \right) dw.$$

Trick 3: $\displaystyle\int_{|w|=1} \frac{f(w)\, \bar{a}}{1 - \bar{a}\, w} \, dw = 0$ by Cauchy's integral theorem.

Conclusion: Cauchy's integral formula on a disk

If f is analytic on a neighborhood of the closed unit disk, and a is an arbitrary point in the open unit disk, then

$$f(a) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w-a}\, dw.$$

Assignment (not to hand in)

Make a list of the main concepts and theorems covered so far (in preparation for the exam coming up on Thursday, October 4).