# Reminder

The first examination takes place in class on Thursday, October 4.

Please bring paper and a writing implement to the exam.

# Recap: Cauchy's integral formula on the unit disk

If $f$ is analytic on a neighborhood of the closed unit disk, then

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{f(w)}{w-z}\,dw=\begin{cases}f(z) & \text{when } |z|<1,\\ 0 & \text{when } |z|>1.\end{cases}$$

## Remarks

1. The symbol $\displaystyle\int_{|w|=1}$ or $\displaystyle\oint_{|w|=1}$ is an abbreviation for $\displaystyle\int_{\gamma}$ where $\gamma(t)=e^{it}$, $0\le t\le 2\pi$.
2. When $|z|=1$, the integral may be divergent.
3. Radius 1 is merely a convenience. Making a dilation shows that a corresponding formula holds when $f$ is analytic on a disk of arbitrary radius.

# Some consequences of Cauchy's integral formula on a disk

- The values of an analytic function $f$ on the boundary determine the values of $f$ everywhere inside the disk.
- Applying Leibniz's rule to differentiate under the integral sign shows that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w|=1} \frac{f(w)}{(w-z)^{n+1}}\, dw, \qquad |z| < 1, \quad n \in \mathbb{N}.$$

  Thus the existence of a continuous first-order complex derivative implies the existence of continuous complex derivatives of all orders!

# Proof of the fundamental theorem of algebra

Suppose, seeking a contradiction, that $P(z)$ is a nonconstant polynomial that is never equal to 0. Then $1/P(z)$ is entire.

Since $|P(z)| \to \infty$ when $|z| \to \infty$, there is a radius $R$ such that

$$\frac{1}{|P(z)|} < \frac{1}{2|P(0)|} \qquad \text{when } |z| = R.$$

Apply Cauchy's integral formula to $1/P(z)$ to see that

$$\left|\frac{1}{P(0)}\right| = \left|\frac{1}{2\pi i}\int_{|z|=R} \frac{1/P(z)}{z-0}\,dz\right| \le \frac{1}{2|P(0)|},$$

contradiction.

# Liouville's theorem for entire functions

Theorem (3.4 on page 77)

*The range of an entire function is either a single point or an unbounded set.*

Proof (different from the one in the book).

If bounded, the range is contained in some disk $B(0; M)$. If $z$ is arbitrary, and $r$ is any radius larger than $|z|$, then

$$|f(z) - f(0)| = \left| \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w - z} - \frac{f(w)}{w - 0} \, dw \right|$$
$$= \left| \frac{z}{2\pi i} \int_{|w|=r} \frac{f(w)}{(w - z)w} \, dw \right| \leq \frac{|z| \, M}{r - |z|}.$$

Let $r \to \infty$ to see that $f(z) \equiv f(0)$. □