Exam follow-up

- Solutions are posted.
- Grading algorithm: $40 + \sum_{k=1}^{6} n_k$, where $0 \leq n_k \leq 10$.
- Class statistics: mean $= 87.5$, median $= 89$, maximum $= 99$.

Remark on Exam Problem 3

To *prove* the Cauchy–Riemann equations in polar coordinates requires a computation.

But to *remember* the equations requires only an example.

The identity function z is analytic, and $z = re^{i\theta}$, so the relation between $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ must be that $r\dfrac{\partial f}{\partial r} = \dfrac{1}{i} \cdot \dfrac{\partial f}{\partial \theta}$.

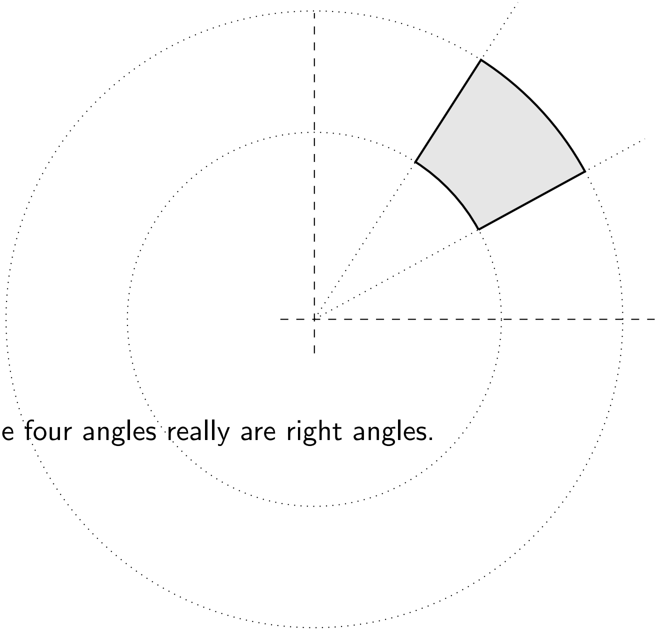
Remark on Exam Problem 4

If $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, and $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers that converges, then $\lim_{n\to\infty} g(a_n) = g\big(\lim_{n\to\infty} a_n\big)$.
[equivalent to the definition of continuity]

But $\limsup_{n\to\infty} g(a_n)$ is not necessarily equal to $g\big(\limsup_{n\to\infty} a_n\big)$.

Counterexample: $g(x) = \dfrac{1}{1+x^2}$, and $a_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$

Remark on Exam Problem 6



The four angles really are right angles.

Recap: Cauchy's integral formula on the unit disk

If f is analytic on a neighborhood of the closed unit disk, then

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{f(w)}{w-z}\,dw = \begin{cases} f(z) & \text{when } |z|<1, \\ 0 & \text{when } |z|>1, \end{cases}$$

$$\frac{n!}{2\pi i}\int_{|w|=1}\frac{f(w)}{(w-z)^{n+1}}\,dw = \begin{cases} f^{(n)}(z) & \text{when } |z|<1, \\ 0 & \text{when } |z|>1. \end{cases}$$

Corollary: Existence of power series expansions

Theorem (2.8 on page 72)

If f is analytic in a disk with center z_0, then f can be represented by a power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ that converges (at least) in that disk. Moreover, $c_n = \frac{1}{n!} f^{(n)}(z_0)$.

Idea of the proof.

Use Cauchy's integral formula to write $f(z)$ as $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(w)}{w-z}\,dw$, where γ is a circle centered at z_0. Express $\dfrac{1}{w-z}$ as $\dfrac{1}{w-z_0}\cdot\dfrac{1}{1-\frac{z-z_0}{w-z_0}}$, expand in a geometric series, and integrate the series term-by-term. $\square$

Example: Bernoulli numbers

Suppose $f(z) = \dfrac{z}{e^z - 1}$ when $z \neq 0$, and 1 when $z = 0$.
Then f is analytic in a neighborhood of 0.
The *Bernoulli number* B_n is $f^{(n)}(0)$, so $f(z) = \sum_{n=0}^{\infty} \dfrac{B_n}{n!} z^n$.
What is the radius of convergence of this power series?
Answer: 2π, the radius of the largest disk in which f is analytic.

It can be shown [Exercise IV.2.14 on page 76] that

$$\tan(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n} (2^{2n} - 1) B_{2n}}{(2n)!} \, z^{2n-1}.$$

What is the radius of convergence of this power series?
Answer: $\pi/2$.

Zeros of analytic functions

Theorem (Corollary 3.10 on page 79)

If f is a nonconstant analytic function on a connected open set, then the zeros of f are isolated.

Proof.

Suppose $f(z_0) = 0$. Consider the Taylor series for f, say $\sum_{n=0}^{\infty} c_n(z - z_0)^n$.

If f is not identically zero, then there is a first nonzero coefficient, say c_k. So $f(z) = (z - z_0)^k \sum_{n=0}^{\infty} c_{n+k}(z - z_0)^n$.

The first factor is nonzero when $z \neq z_0$. The second factor is nonzero when $z = z_0$, so by continuity, nonzero in a neighborhood of z_0. So the zeros of f cannot accumulate at z_0. $\square$

Assignment due next time

- Exercise 5 in section IV.2 on page 74, which asks for the Taylor series in powers of $(z - i)$ for the principal branch of the logarithm, and the radius of convergence of the series.
- Exercise 8 in section IV.3 on page 80, which says that the analytic functions on a region (connected open set) form an integral domain.