Warm-up exercise on zeros of analytic functions

Suppose G is a connected open subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is a nonconstant analytic function. Prove:

(a) If K is a compact subset of G, then the number of zeros of f in K is finite.

(b) The set of zeros of f in G is countable (can be put in one-to-one correspondence with a subset of $\mathbb{N}$).

Application of isolated zeros: Persistence of functional relations

You know that $e^{a+b} = e^a e^b$ when $a \in \mathbb{R}$ and $b \in \mathbb{R}$.
Why is $e^{z+w} = e^z e^w$ when $z \in \mathbb{C}$ and $w \in \mathbb{C}$?

Proof.

Suppose $f_b(z) = e^{z+b} - e^z e^b$ when $b \in \mathbb{R}$.
Then f_b is entire (analytic in the entire plane), and $f_b(a) = 0$ when $a \in \mathbb{R}$. The zeros of f_b are not isolated, so f_b is identically zero.

Next suppose $g_z(w) = e^{z+w} - e^z e^w$ when $z \in \mathbb{C}$.
Then g_z is entire, and $g_z(b) = 0$ when $b \in \mathbb{R}$. The zeros of g are not isolated, so g_z is identically zero. $\square$

Are the points inside or outside?

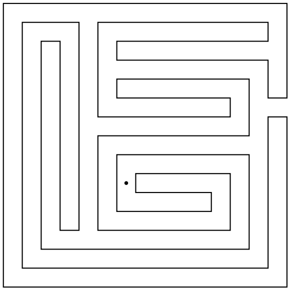

Figure A

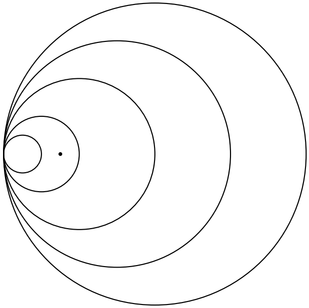

Figure B

Assignment due next time

- Solve Exercise 10 in §3 of Chapter IV (page 80).
- Solve Problem 3 on the August 2008 qualifying examination.