Jordan curve theorem

Theorem (Jordan, 1887)

If J is the homeomorphic image of a circle (a simple closed curve), then $\mathbb{C} \setminus J$ has precisely two connected components, one bounded and one unbounded.

Proof.

Beyond the scope of this course; usually proved in a course on algebraic topology as the low-dimensional case of the more general Jordan–Brouwer separation theorem. $\square$

Camille Jordan (1838–1922), French;
L. E. J. Brouwer (1881–1966), Dutch.

Winding number $n(\gamma; b)$, or index $\mathrm{Ind}(\gamma, b)$

Theorem (Proposition 4.1 on page 81)

When γ is a **closed** *rectifiable path, and b is a point not on the image of γ, the value of $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{1}{z-b}\,dz$ is an* **integer***.*

The intuitive proof (compare page 82).

Think of $\dfrac{1}{z-b}\,dz$ as $d\log(z-b)$. So $\displaystyle\int_\gamma \frac{1}{z-b}\,dz$ represents the change in $\log(z-b)$ along the path γ.

Although $\ln|z-b| + i\arg(z-b)$ may not be well defined globally, there are locally defined branches at each point of γ, and the *change* is independent of the choice of branch.

There is no net change in $\ln|z-b|$ around the closed path γ, so the integral equals i times the net change of $\arg(z-b)$ around γ, or i times $2\pi n$ for some integer n. $\square$

The "homology form" of Cauchy's integral formula

Theorem (5.6 on page 85)

Suppose G is an open subset of $\mathbb{C}$, and $f\colon G \to \mathbb{C}$ is analytic, and γ is a closed rectifiable path in G such that for every point b in $\mathbb{C} \setminus G$, the winding number of γ about b equals 0.
[The winding-number hypothesis is vacuous if G has no holes.]
If $w \in G$, and w does not lie on the image of γ, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz = f(w) n(\gamma; w).$$

More generally, $\displaystyle\sum_{j=1}^{k} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z-w}\, dz = \sum_{j=1}^{k} f(w) n(\gamma_j; w)$ *if* $\displaystyle\sum_{j=1}^{k} n(\gamma_j; b) = 0$ *for every point b in $\mathbb{C} \setminus G$, and $w \in G$, and w does not lie on any of the closed rectifiable curves $\gamma_1, \ldots, \gamma_k$ in G.*

A remark on Dixon's 1971 proof of Cauchy's formula

(to be continued)

Assignment due next time

(A)
- Read on page 73 about Cauchy's estimate for derivatives.
- Solve Exercise 1 on page 80.

(B) Solve Exercise 4 on page 83.