A remark on Dixon's 1971 proof of Cauchy's formula

If $g(z, w) := \dfrac{f(z) - f(w)}{z - w}$ when $z \neq w$, and $g(w, w) = f'(w)$, why is $g\colon D \times D \to \mathbb{C}$ jointly continuous and analytic?

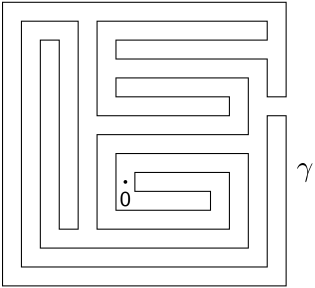
Away from "the diagonal" $\{\, (z, w) \in D \times D : z = w \,\}$, the function g equals an analytic function divided by a nonzero analytic function, so g is analytic.

What if z and w are close to the same value b, say $|z - b| < \varepsilon$ and $|w - b| < \varepsilon$? Cauchy's formula for a disk implies that

$$\begin{aligned} g(z, w) &= \frac{1}{z - w} \cdot \frac{1}{2\pi i} \int_{|\zeta - b| = \varepsilon} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta - b| = \varepsilon} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)}\, d\zeta, \end{aligned}$$

and the final integral is analytic in z and w by Leibniz's rule.

Example



$$\int_\gamma \frac{e^z}{z}\,dz = \,?$$

Solution: If $G = \mathbb{C} \setminus \{0\}$ and $f(z) = e^z/z$, then the homology version of Cauchy's theorem implies that

$$\int\limits_{\substack{\gamma \\ \text{counter-} \\ \text{clockwise}}} f(z)\,dz + \int\limits_{\substack{\text{clockwise} \\ \text{circle} \\ \text{about } 0}} f(z)\,dz = 0.$$

Therefore $\displaystyle\int_\gamma \frac{e^z}{z}\,dz = \int_{|z|=1} \frac{e^z}{z}\,dz = 2\pi i\, e^0 = 2\pi i.$

A converse to Cauchy's theorem

Theorem (homology version of Cauchy's theorem)

If $f\colon G \to \mathbb{C}$ *is analytic, then* $\int_\gamma f(z)\,dz = 0$ *for every closed curve* γ *having winding number zero about points in* $\mathbb{C} \setminus G$.

Theorem (Giacinto Morera, 1886)

If $f\colon G \to \mathbb{C}$ *is a continuous function, and* $\int_\gamma f(z)\,dz = 0$ *for every closed curve* γ*, then* f *is analytic.*

Assignment due next time

1. Exercise 2 in §4 of Chapter IV, which asks for a closed rectifiable curve whose winding number about different points takes every integer value.

   (Remember that by definition, the domain of a curve is supposed to be a closed and bounded interval in $\mathbb{R}$.)

2. Exercise 8 in §5 of Chapter IV: the limit of a uniformly convergent sequence of analytic functions is analytic.

   (For reminders about uniform convergence, see §6 of Chapter II. You may use the knowledge from real analysis that if $\{g_n\}$ is a sequence of continuous functions converging uniformly to a limit function g on a compact interval $[a, b]$, then g is continuous on $[a, b]$, and $\int_a^b g_n(t)\,dt$ converges to $\int_a^b g(t)\,dt$.)