A converse to Cauchy's theorem

Theorem (homology version of Cauchy's theorem)

If $f\colon G \to \mathbb{C}$ *is analytic, then* $\displaystyle\int_\gamma f(z)\,dz = 0$ *for every closed curve* γ *having winding number zero about points in* $\mathbb{C} \setminus G$*.*

Theorem (Giacinto Morera, 1886)

If $f\colon G \to \mathbb{C}$ *is a continuous function, and* $\displaystyle\int_\gamma f(z)\,dz = 0$ *for every closed curve* γ*, then* f *is analytic.*

Remark

It suffices to know that $\displaystyle\int_\gamma f(z)\,dz = 0$ for sufficiently many curves, say all small triangles [5.10 in Chapter IV] or all small rectangles.

Proof of Morera's theorem

Analyticity is a *local* property, so assume without loss of generality that the domain G is a disk with center z_0.

Define $F(z) := \int_{z_0}^{z} f(w)\,dw$, where $\int_{z_0}^{z}$ means $\int_{\gamma_z}$ for some path γ_z joining z_0 to z.

The integral is well defined (independent of the choice of γ_z) by the hypothesis that integrals over *closed* curves equal zero.

The fundamental theorem of calculus implies that $F'(z)$ exists and equals $f(z)$.

So F is analytic, and therefore its derivative f is analytic.

The path-deformation principle (IV.6.13)

If $f\colon G \to \mathbb{C}$ is analytic, and γ_0 and γ_1 are two rectifiable paths in G joining a point z_0 to a point z_1, and γ_0 can be continuously deformed to γ_1 within G, then $\displaystyle\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.

The intuition

The *closed* path consisting of γ_0 followed by γ_1 backward should have zero winding number around points in $\mathbb{C} \setminus G$, so the statement should follow from the homology version of Cauchy's theorem.

When G is a disk, we already know that this intuition is valid. For more general regions, a proof is needed.

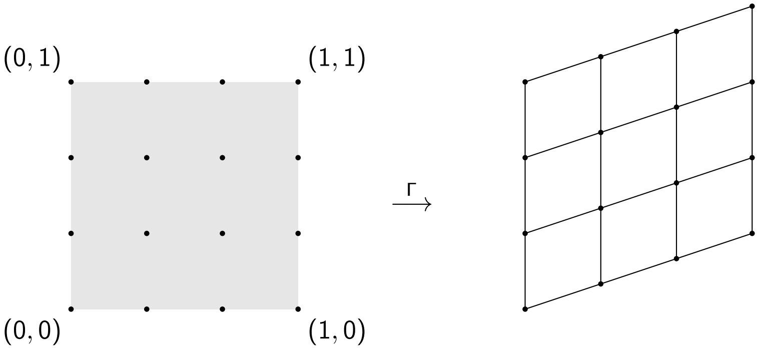
The precise meaning of “continuous deformation”

Paths $\gamma_0 \colon [0,1] \to G$ and $\gamma_1 \colon [0,1] \to G$ joining z_0 to z_1 are (fixed endpoint) *homotopic* in G if there exists a continuous function $\Gamma \colon [0,1] \times [0,1] \to G$ such that
$\Gamma(s,0) = \gamma_0(s)$ and $\Gamma(s,1) = \gamma_1(s)$
(top and bottom edges of the square) and
$\Gamma(0,t) = z_0$ and $\Gamma(1,t) = z_1$
(left and right sides of the square).

A technical issue for proving the path-deformation principle

The intermediate continuous paths $s \mapsto \Gamma(s,t)$ need not be rectifiable when $0 < t < 1$, even though γ_0 and γ_1 are rectifiable.

Sketch of proof of path-deformation principle



(The picture is fake, because Γ is not injective.) Connect the nodes in the image of Γ by line segments to make a network of "quadrilaterals," each oriented counterclockwise. If the mesh is fine enough, each quadrilateral is contained in a disk inside G, so the integral over each quadrilateral equals 0. Add the integrals up.

Assignment due next time

1. Exercise 4 in §6 of Chapter IV, which says that every closed curve in the punctured plane is homotopic to a closed curve whose image lies on the unit circle.

2. Exercise 11 in §6 of Chapter IV, which asks for the value of $\int_\gamma \frac{e^z - e^{-z}}{z^4}\, dz$ for three specified curves.