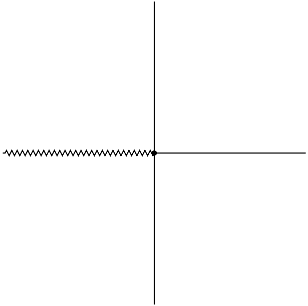
Follow-up on winding numbers

Two of you found an explicit parametrization online of a curve that has every integer winding number.

A calculation is needed to verify that this curve is rectifiable.

Warm-up exercise

If log is the principal branch of the logarithm, is it correct to say that $\log(z^2) = 2\log(z)$?



Answer: OK if z lies in the right-hand half-plane, but not otherwise.

Simple connectivity

An open set G in $\mathbb{C}$ is *simply connected* if any of the following equivalent properties holds.

- Every two paths in G having the same endpoints are fixed-endpoint homotopic.
- Every closed path in G is homotopic (through closed paths) to a constant path.
- There are no holes in G.
- The complement of G in $\mathbb{C}_\infty$ is connected.

Remarks

- More equivalences are stated in §2 of Chapter VIII.
- Many authors require a simply connected set additionally to be (path) connected.

Logarithms of functions

A function g is a *logarithm* of a function f in a region G if $e^{g(z)} = f(z)$ for every z in G.

Theorem (6.17 in Chapter IV)

If $f\colon G \to \mathbb{C}$ is an analytic function without zeros, and G is simply connected, then f has an analytic logarithm.

Proof.

Fix a point z_0 in G, choose any value of $\log(f(z_0))$, and define $g(z)$ to be $\log(f(z_0)) + \displaystyle\int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw$. The integral is well defined (independent of the integration path) because G is simply connected.

Compute that the derivative of $f(z)e^{-g(z)}$ is identically zero to conclude that g is a logarithm of f in G. □

A subtlety about logarithms of functions

A natural notation for a logarithm of a function $f(z)$ is $\log f(z)$, but this notation must be handled with care, for the expression typically is *not* a composite function.

Warm-up exercise revisited

If G is the plane with the origin and the negative real numbers removed, then z^2 is never equal to 0 on G, so there is an analytic logarithm of z^2: namely, $2\log(z)$.

But this function cannot be written as $\log(z^2)$.

Exercise

If $f(z) = (z-1)(z-2)\cdots(z-617)$, then $\dfrac{f'(z)}{f(z)} = ?$

Answer: $\dfrac{1}{z-1} + \dfrac{1}{z-2} + \cdots + \dfrac{1}{z-617}$

Assignment due next time

1. Exercise 5 in §6 of Chapter IV, which asks for the value of $\int_\gamma \frac{1}{z^2+1}\, dz$ for a certain curve γ.
2. Exercise 10 in §6 of Chapter IV, which asks for all possible values of $\int_\gamma \frac{1}{z^2+1}\, dz$ as the closed curve γ varies.

Hint for both exercises: $\frac{1}{z^2+1} = \frac{i/2}{z+i} + \frac{-i/2}{z-i}$
(via "partial fractions").
Remember that the integral theorems you know have a factor $2\pi i$ somewhere.