Counting zeros

Theorem (basic version of the “argument principle”)

If G is a simply connected region, and γ is a simple closed rectifiable curve in G oriented counterclockwise, and $f : G \to \mathbb{C}$ is analytic, and f has no zeros on (the image of) γ, then

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz &= \textit{number of zeros of } f \textit{ inside } \gamma \\ &\qquad \textit{(counted according to multiplicity)} \\ &= \frac{1}{2\pi}\textit{net change in } \arg f(z) \textit{ around } \gamma.\end{aligned}$$

For generalizations, see Theorem 7.2 in Chapter IV and §3 of Chapter V.

Example

Problem 2 on the August 2018 qualifying exam asks, "How many zeros of the polynomial $z^4 + 3z^2 + z + 1$ lie in the right half-plane?"

Solution

There are no zeros on either coordinate axis.

The polynomial has real coefficients, so the zeros occur in complex-conjugate pairs.

Apply the argument principle on a quarter circle in the first quadrant with large radius to see that there is one zero in the first quadrant.

Rouché's theorem about walking a dog

Theorem (Rouché, 1862)

If G is a simply connected region, and γ is a simple closed rectifiable curve in G oriented counterclockwise, and f and φ are analytic functions on G, and

$$|\varphi(z)| < |f(z)| \qquad \text{when } z \text{ is on } \gamma,$$

then f and $f \pm \varphi$ have the same number of zeros inside γ.

Example

Problem 10 on the January 2018 qualifying exam says that if $1 < a$, then the function $e^z - z - a$ has exactly one zero in the half-plane where $\mathrm{Re}(z) < 0$.

Solution: take $f(z) = z + a$ and $\varphi(z) = e^z$ and apply the theorem on a big semi-circle.

Assignment due next time

1. Solve Problem 9 on the August 2016 qualifying exam, which asks for the number of zeros of the function
$$z^8 + \exp(2016\pi z)$$
in the half-plane where $\mathrm{Re}(z) < 0$.

2. Solve Problem 3 on the January 2017 qualifying exam, which says that the partial sums of the Maclaurin series of the exponential function are eventually zero-free on a fixed disk.