Exam 2 follow-up

- Solutions are posted.
- Grading algorithm: $40 + \sum_{k=1}^{6} n_k$, where $0 \leq n_k \leq 10$.
- Class statistics: mean $= 85.5$, median $= 88$, maximum $= 100$.

Classification of isolated singularities

If f is analytic on a punctured disk $\{ z \in \mathbb{C} : 0 < |z - b| < r \}$, then either

1. $|f|$ is bounded near b, in which case f has a *removable singularity* (f extends to be analytic in the whole disk); or
2. $\lim_{z \to b} |f(z)| = \infty$, in which case f has a *pole*; or
3. neither of the above: there is some sequence tending to b along which $|f| \to \infty$ and some other sequence along which $|f|$ remains bounded; f has an *essential singularity*.

Examples (when $b = 0$)

1. $\dfrac{\sin(z)}{z}$
2. $\dfrac{1}{z^2}$
3. $e^{1/z}$

Riemann's theorem on removable singularities

Hypothesis: f is analytic and *bounded* near an isolated singularity.
Conclusion: The singularity is removable.

Proof (different from the proof in the book, V.1.2).

WLOG, suppose f is analytic and bounded when $0 < |z| \leq 1$. If

$$g(z) := \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w - z} \, dw,$$

then $g(z)$ is analytic when $|z| < 1$. By the homology form of Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \left(\int_{|w|=1} - \int_{|w|=\varepsilon} \right) \frac{f(w)}{w - z} \, dw \qquad \text{when } \varepsilon < |z| < 1.$$

Holding z fixed, let $\varepsilon \to 0$ to see that $f(z) = g(z)$ when $z \neq 0$. □

The order of a pole

If $\lim_{z\to b} |f(z)| = \infty$, then $\lim_{z\to b} \frac{1}{f(z)} = 0$.

So if f has a pole at b, then $\frac{1}{f}$ has a removable singularity at b.

If $\frac{1}{f}$ has a zero of order m at b, then f has a *pole of order* m.

Equivalently, m is the (smallest) positive integer such that $\lim_{z\to b}(z-b)^m f(z)$ exists and is nonzero.

The range of f near an essential singularity

Theorem (Picard's great theorem)

If f has an essential singularity at b, then for every (small) positive δ, the restriction of f to $\{ z \in \mathbb{C} : 0 < |z - b| < \delta \}$ has range equal to either $\mathbb{C}$ or $\mathbb{C} \setminus \{\text{one point}\}$.

Therefore a function takes every complex value—with one possible exception—infinitely often in every punctured neighborhood of an essential singularity.

Examples

- $z^2 \, e^{1/(z-b)}$ has an essential singularity at b and exceptional value 0.
- $\sin(1/z)$ has an essential singularity at 0 and no exceptional value.

Assignment due next time

- Exercise 6 in Section 1 of Chapter V.
- Exercise 13(a) in Section 1 of Chapter V.