Classification of isolated singularities, revisited

View a punctured disk as a degenerate annulus
$\{ z \in \mathbb{C} : 0 < |z - b| < r \}$.

A function analytic in a punctured disk can be represented by a Laurent series.

1. The isolated singularity at the center of the disk is a removable singularity if and only if the Laurent series is a Taylor series: there are no terms having negative exponent.
2. The isolated singularity is a pole if and only if there are finitely many terms (at least one) in the Laurent series having negative exponent.
3. The isolated singularity is an essential singularity if and only if there are infinitely many terms in the Laurent series having negative exponent.

Definition of residue (Cauchy, 1826)

Motivation: If $f(z) = \sum_{n=0}^{\infty} c_n(z-b)^n$ (a Taylor series in a disk centered at b), then the coefficient c_1 is special: it represents the value of the derivative $f'(b)$.

If $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-b)^n$ (a Laurent series in a punctured disk centered at b), then the coefficient c_{-1} is special: it represents the value of the integral $\dfrac{1}{2\pi i}\displaystyle\int_{|z-b|=\varepsilon} f(z)\,dz$ around a small circle.

Definition: This number is the *residue* of f at the isolated singularity b.

Basic version of the residue theorem

If G is a simply connected open set, and if f has only isolated singularities in G, and if γ is a simple closed rectifiable curve in G not passing through any singular points, then

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \text{sum of residues of } f \text{ at singularities inside } \gamma.$$

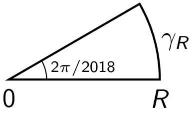
Proof.

Use the homology form of Cauchy's theorem to replace γ with a finite number of small circles, one around each singular point. $\square$

Theorem V.2.2 is a version with winding numbers.

Sample application of the residue theorem

$$\int_0^\infty \frac{x^{20}}{1+x^{2018}}\,dx = \frac{\pi}{2018\sin(\frac{21\pi}{2018})}$$



Solution

$$\int_{\gamma_R} \frac{z^{20}}{1+z^{2018}}\,dz = 2\pi i(\text{residue at } e^{\pi i/2018}) = \frac{-2\pi i\, e^{21\pi i/2018}}{2018}$$

$$= \int_0^R \frac{x^{20}}{1+x^{2018}}\,dx - e^{42\pi i/2018}\int_0^R \frac{x^{20}}{1+x^{2018}}\,dx + O(R^{21-2018})$$

so $$\int_0^\infty \frac{x^{20}}{1+x^{2018}}\,dx = \frac{-2\pi i\, e^{21\pi i/2018}}{2018(1-e^{42\pi i/2018})}.$$

Assignment

- Travel safely along a closed, piecewise smooth path.