Reminders

- The final class meeting is December 4
  (Tuesday, redefined as Thursday).
- The final examination takes place in this room on the morning of Wednesday, 12 December, 8:00–10:00.

Maximum-modulus theorem

- If $f(z)$ is analytic on a connected open set, then $|f(z)|$ cannot attain a local maximum unless f is a constant function.
  Proof: immediate from open-mapping theorem.
- If $f(z)$ is analytic on a *bounded* open set and continuous on the closure, then $\max |f(z)|$ is attained at some boundary point.
  Proof: $|f(z)|$ is a continuous real-valued function on a compact set, so attains a maximum somewhere; not in the interior, except in trivial cases.
- In the preceding statement, boundedness of the domain is essential: consider e^z on the right-hand half-plane.
- Theorem VI.1.1.4 is a fancy version with continuity of f and boundedness of the domain replaced by a $\limsup$ condition.

Application of the maximum-modulus theorem

Theorem (the Schwarz lemma, VI.2.2.1)

If f (analytic) maps the unit disk into itself, fixing the origin, then $|f(z)| \leq |z|$ when $|z| < 1$.

Proof.

By hypothesis, $\dfrac{f(z)}{z}$ has a removable singularity at 0.
Apply the maximum-modulus theorem on a disk of radius r less than 1 to deduce that $\left|\dfrac{f(z)}{z}\right| \leq \dfrac{1}{r}$ when $|z| \leq r$. Let $r \to 1$. □

Remarks

- If equality holds, even for one nonzero value of z, then f must be a rotation.
- Cauchy's estimate for derivatives implies that $|f'(0)| \leq 1$, even if $f(0) \neq 0$.

Assignment

- If you have not already done so, please complete the online course evaluation.
- Find an exercise in the textbook that you do not know how to solve.