Examination 1

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Find all values of the complex variable z for which

$$(\operatorname{Re}(z))^4 = \operatorname{Re}(z^4).$$

Solution. Method 1. Write the complex variable z as $x + iy$, where x and y are real. The equation reduces to the following:

$$x^4 = \operatorname{Re}\left(x^4 + 4x^3 iy + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4\right) = x^4 - 6x^2y^2 + y^4,$$

or $6x^2y^2 = y^4$. This equation holds when $y = 0$ and when $6x^2 = y^2$. Thus the solution set is the union of three lines: the real axis, the line where $y = \sqrt{6}\,x$, and the line where $y = -\sqrt{6}\,x$.

Method 2. The equation certainly holds when $z = 0$. When $z \neq 0$, write z as $re^{i\theta}$ in polar coordinates and divide the equation by r^4 to see that

$$\left(\operatorname{Re}(e^{i\theta})\right)^4 = \operatorname{Re}(e^{4i\theta}), \qquad \text{or} \qquad (\cos(\theta))^4 = \cos(4\theta).$$

A convenient way to expand the right-hand side as a polynomial in $\cos(\theta)$ is to apply the double-angle trigonometric identity twice:

$$\cos(4\theta) = 2\big(\cos(2\theta)\big)^2 - 1 = 2\big(2(\cos(\theta))^2 - 1\big)^2 - 1 = 8(\cos(\theta))^4 - 8(\cos(\theta))^2 + 1.$$

Thus $7(\cos(\theta))^4 - 8(\cos(\theta))^2 + 1 = 0$. By the quadratic formula,

$$(\cos(\theta))^2 = \frac{8 \pm \sqrt{64 - 28}}{14} = \quad 1 \quad \text{or} \quad \frac{1}{7}.$$

If $(\cos(\theta))^2 = 1$, then θ is 0 or π, so z lies on the real axis. If $(\cos(\theta))^2 = 1/7$, then $(\sin(\theta))^2 = 6/7$, so $(\tan(\theta))^2 = 6$, and z lies on a line through the origin with slope $\pm\sqrt{6}$. Thus the solution set consists of three lines, the same three lines found by the first method.

2. When the letter z represents a complex variable, is it valid to say that $\lim_{z\to\infty}(z + \overline{z}) = \infty$? Explain why or why not.

Solution. Tending to infinity in the complex world means eventually escaping from every compact set; equivalently, the absolute value should tend to infinity. So the equation is claiming that $|z + \overline{z}|$ gets arbitrarily large when $|z|$ gets arbitrarily large. But if $z = iy$, where y is a real number, then $z + \overline{z} = 0$, no matter how large y is. So the claim is not valid.

Remark. The imaginary axis is the only line through the origin along which the expression $z + \overline{z}$ fails to tend to infinity.

3. Suppose f is an analytic function on $\mathbb{C} \setminus \{0\}$, and the real part of $f(z)$ equals $\dfrac{\sin(2\theta)}{r^2}$ in standard polar coordinates. (As usual, $r = |z|$, and $\theta = \arg(z)$.) Find a concrete expression for f as a function of the variable z.

Solution. Method 1. The solution can be found by the "guess and check" method. The angle doubling suggests that f should be related to the squaring function. The presence of r^2 in the denominator suggests that an inversion is involved too. Accordingly, it is natural to examine the expression for $1/z^2$ in polar coordinates:

$$\frac{1}{z^2} = \frac{1}{r^2 e^{2i\theta}} = \frac{1}{r^2} e^{-2i\theta} = \frac{1}{r^2}(\cos(2\theta) - i\sin(2\theta)).$$

This expression reveals that $\dfrac{i}{z^2}$ is a solution for $f(z)$.

The real part of f does not uniquely determine f, however. The most general expression for $f(z)$ is $\dfrac{i}{z^2} + ic$, where c is an arbitrary real constant.

Method 2. There is a systematic way to determine f via the Cauchy–Riemann equations. Suppose $f = u + iv$, where u and v are real-valued functions. From a homework exercise, you know the Cauchy–Riemann equations in polar coordinates:

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

The given information is that $u = \dfrac{\sin(2\theta)}{r^2}$, so $\dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r} = \dfrac{-2\sin(2\theta)}{r^2}$. Integrating with respect to θ reveals that $v = \dfrac{\cos(2\theta)}{r^2} + g(r)$ for some real-valued function g depending only on r.

To determine $g(r)$, observe that

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\sin(2\theta)}{r^2}\right) = \frac{2\cos(2\theta)}{r^2},$$

but also

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} = -r\frac{\partial}{\partial r}\left(\frac{\cos(2\theta)}{r^2} + g(r)\right) = \frac{2\cos(2\theta)}{r^2} - rg'(r).$$

Comparing the two expressions for $\dfrac{\partial u}{\partial \theta}$ shows that $-rg'(r) = 0$, so $g'(r) = 0$, whence $g(r)$ is a real constant, say c.

Thus $v = \dfrac{\cos(2\theta)}{r^2} + c$, so $f = u + iv = \dfrac{\sin(2\theta) + i\cos(2\theta)}{r^2} + ic = \dfrac{ie^{-2i\theta}}{r^2} + ic$. The result is the same as before, $f(z) = \dfrac{i}{z^2} + ic$.

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4. Suppose that c_n is a complex number for each natural number n, and the power series $\sum_{n=1}^{\infty} c_n z^n$ has radius of convergence equal to 4. What can you say about the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n^2 z^n$? Explain how you know.

Solution. To see what the answer should be, test a simple example. The geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n} z^n$ evidently has radius of convergence equal to 4. The geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{4^n}\right)^2 z^n$ has radius of convergence equal to 16.

What is needed now is a proof that this example generalizes: namely, the series $\sum_{n=1}^{\infty} c_n^2 z^n$ always has radius of convergence equal to 16.

Method 1. The proof from class of the theorem asserting the existence of the radius of convergence shows that this radius equals what could be called the "radius of boundedness." In other words, if r denotes a nonnegative real number, then the radius of convergence of a power series $\sum_{n=1}^{\infty} b_n z^n$ equals the supremum of values of r for which the sequence $\{b_n r^n\}_{n=1}^{\infty}$ is bounded. The reason is that if this sequence is bounded, then comparison with a geometric series shows that the series $\sum_{n=1}^{\infty} b_n z^n$ converges (absolutely) when $|z| < r$; and, conversely, if the series $\sum_{n=1}^{\infty} b_n z^n$ converges for a certain value of z, then the sequence $\{b_n r^n\}_{n=1}^{\infty}$ is bounded when $r \le |z|$.

The sequence $\{c_n r^n\}_{n=1}^{\infty}$ evidently is bounded if and only if the sequence $\{c_n^2 r^{2n}\}_{n=1}^{\infty}$ is bounded. The supremum of values of R for which the sequence $\{c_n^2 R^n\}_{n=1}^{\infty}$ is bounded therefore equals the square of the supremum of values of r for which the sequence $\{c_n r^n\}_{n=1}^{\infty}$ is bounded. In other words, the radius of convergence of the series $\sum_{n=1}^{\infty} c_n^2 z^n$ equals the square of the radius of convergence of the series $\sum_{n=1}^{\infty} c_n z^n$.

Method 2. Cauchy's formula for the radius of convergence of a power series implies that $\limsup_{n\to\infty} |c_n|^{1/n} = 1/4$, and the goal is to prove that $\limsup_{n\to\infty} |c_n^2|^{1/n} = 1/16$. The problem reduces to proving the following statement.

Lemma. *If $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of nonnegative real numbers, then* $\limsup_{n\to\infty}(a_n^2) = \left(\limsup_{n\to\infty} a_n\right)^2$.

Only a little extra work is needed to prove the following more general statement.

Lemma. *If $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of nonnegative real numbers, and $g : [0, \infty) \to [0, \infty)$ is a continuous, weakly increasing function, then* $\limsup_{n\to\infty} g(a_n) = g\left(\limsup_{n\to\infty} a_n\right)$.

Proof. The lim sup is the largest limit of convergent subsequences. So suppose $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers such that $\lim_{k\to\infty} a_{n_k}$ exists and equals $\limsup_{n\to\infty} a_n$. Since g is continuous, $\lim_{k\to\infty} g(a_{n_k})$ exists and equals $g\bigl(\limsup_{n\to\infty} a_n\bigr)$. In principle, there might be another subsequence along which g approaches a larger limit, so what can be concluded at this point is that $\limsup_{n\to\infty} g(a_n) \ge g\bigl(\limsup_{n\to\infty} a_n\bigr)$.

Next suppose $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers such that $\lim_{k\to\infty} g(a_{n_k})$ exists and equals $\limsup_{n\to\infty} g(a_n)$. If $\{n_{k_j}\}_{j=1}^{\infty}$ is a subsequence of $\{n_k\}_{k=1}^{\infty}$ such that the sequence $\{a_{n_{k_j}}\}_{j=1}^{\infty}$ converges, necessarily to a limit less than or equal to $\limsup_{n\to\infty} a_n$, then the continuity and monotonicity of g imply that

$$\limsup_{n\to\infty} g(a_n) = \lim_{k\to\infty} g(a_{n_k}) = \lim_{j\to\infty} g(a_{n_{k_j}}) \le g\bigl(\limsup_{n\to\infty} a_n\bigr).$$

The required equality results from combining the inequalities proved in the preceding two paragraphs. $\square$

Remark. Further generalizations are possible, but you must not try to generalize too far. The lim sup of a product of two sequences is not always equal to the product of the lim sups: consider the two sequences 0, 1, 0, 1, … and 1, 0, 1, 0, … .

5. Suppose r is a positive real number, and $\gamma_r(t) = re^{it}$ when $0 \le t \le \pi$. (This path is a half circle in the upper half-plane.) Is the path integral $\int_{\gamma_r} \frac{1}{z}\,dz$ independent of the value of r? Explain why or why not.

Solution. Method 1. Evaluate the path integral by using the definition and the observation that $\gamma_r'(t) = i\gamma(t)$:

$$\int_{\gamma_r} \frac{1}{z}\,dz = \int_0^{\pi} \frac{1}{re^{it}}\, ire^{it}\,dt = \int_0^{\pi} i\,dt = i\pi.$$

The integral is independent of r.

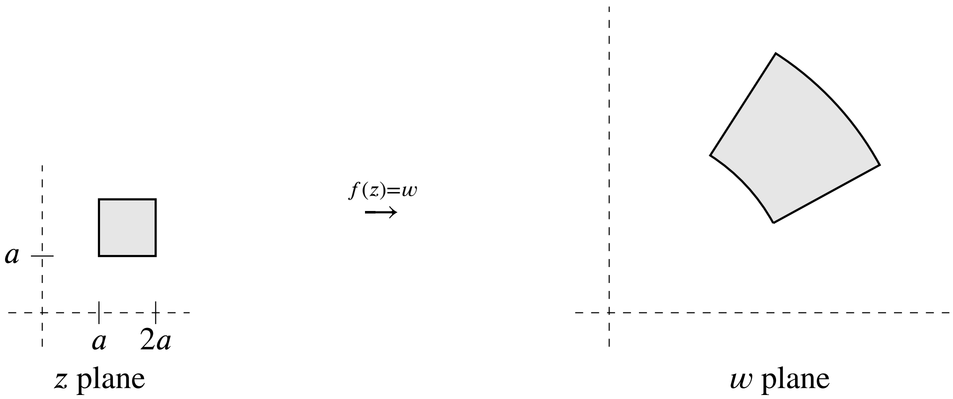
Method 2. Choose a branch of $\log(z)$ for which $-\pi/2 < \arg(z) < 3\pi/2$. In other words, place a branch cut along the bottom half of the imaginary axis. This branch of $\log(z)$ is well defined and analytic on an open set containing the image of γ_r, so

$$\int_{\gamma_r} \frac{1}{z}\,dz = \log(z)\Big|_{\text{endpoints}} = \log(re^{i\pi}) - \log(re^{i0}) = (\ln(r) + i\pi) - (\ln(r) + i0) = i\pi.$$

The integral is independent of r.

Method 3. Suppose $\Gamma_r(t) = re^{it}$ when $-\pi \le t \le \pi$. The path Γ_r is a full circle, and you know from class that $\int_{\Gamma_r} \frac{1}{z}\,dz$ is independent of r. By symmetry, $\int_{\Gamma_r} \frac{1}{z}\,dz = 2\int_{\gamma_r} \frac{1}{z}\,dz$, so $\int_{\gamma_r} \frac{1}{z}\,dz$ must be independent of r.

6. The diagram shows a mapping of a square by some analytic function f. (The dashed lines represent the coordinate axes.) Assuming that the value of a is chosen suitably, can $f(z)$ be equal to $1/z$? or z^2 ? or e^z ? or must $f(z)$ be something else? Explain how you know.



Solution. Since $\arg(1/z) = -\arg(z)$, an inversion will map the first quadrant to the fourth quadrant. Therefore the mapping cannot be inversion.

The lower left-hand corner of the domain square is the point $a(1+i)$. The squaring mapping takes this point to $2ai$ on the imaginary axis. So the mapping cannot be z^2.

Since $e^z = e^x e^{iy}$, the exponential map takes horizontal line segments (where y is constant) to segments of rays with constant angle. And the exponential mapping takes vertical line segments (where x is constant) to circular arcs. This description appears to fit the picture in the range if a has a suitable value. (Indeed, I created the picture from the exponential map by taking a equal to $1/2$.)

Remark. The mapping could be something else, for there are infinitely many holomorphic bijections between the indicated regions. Given one such mapping, you can compose with a holomorphic automorphism of either region to get another such mapping. You will learn later in the course that both regions have transitive groups of holomorphic automorphisms. And if you allow mappings that are surjective but not injective, then there are even more possibilities.