Examination 1

1. Suppose $u(x, y)$ is a twice continuously differentiable real-valued function on an open subset of the plane. Show that u is harmonic if and only if $\partial u/\partial z$ is holomorphic.

   Hint: Recall that a harmonic function is one for which $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, and the operator $\dfrac{\partial}{\partial z}$ is an abbreviation for $\dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$.

   Solution. By hypothesis, the function $\partial u/\partial z$ has continuous first-order (real) partial derivatives, so this function is holomorphic if and only if the Cauchy–Riemann equations hold for the real part and the imaginary part of $\partial u/\partial z$. Write

$$\frac{\partial u}{\partial z} = U + iV, \qquad \text{where} \qquad U = \frac{1}{2}\cdot\frac{\partial u}{\partial x} \quad \text{and} \quad V = -\frac{1}{2}\cdot\frac{\partial u}{\partial y}.$$

   One Cauchy–Riemann equation says that

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x},$$

   and this equation holds automatically by the equality of the mixed second-order partial derivatives of u. The other Cauchy–Riemann equation says that

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y},$$

   which is equivalent to saying that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2},$$

   in other words, that u is harmonic.

   An alternative solution, using Wirtinger's notation, is to observe that

$$\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

   since the mixed second-order partial derivatives cancel. Accordingly, the function u is harmonic if and only if

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right) = 0,$$

   which is equivalent to saying that $\partial u/\partial z$ is holomorphic.

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2. Suppose $f(z)$ is a holomorphic function whose real part is $u(x, y)$ and whose imaginary part is $v(x, y)$. Show that the gradient vector of the function u and the gradient vector of the function v are orthogonal to each other.
(This problem says—in the language of real calculus—that the level curves of u and the level curves of v are families of orthogonal trajectories.)

Solution. The gradient vectors are

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \qquad \text{and} \qquad \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right),$$

and the scalar product of these two vectors equals

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}.$$

The second summand is the negative of the first, for the Cauchy–Riemann equations say that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$. Accordingly, the two gradient vectors are orthogonal, since the scalar product equals zero.

3. Let (p_n) denote the sequence of prime numbers: namely, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, and so forth. For which values of the complex number z does the infinite series $\sum_{n=1}^{\infty} z^{p_n}$ converge? Explain how you know.

Solution. If $|z| = 1$, then $|z^{p_n}| = 1$ for every n. Since the terms of the series do not tend to 0 when n increases, the infinite series diverges when $|z| = 1$, and a fortiori diverges when $|z| > 1$. On the other hand, the terms of the series remain bounded when $|z| = 1$, so the "radius of boundedness" (which equals the radius of convergence) is no smaller than 1. In summary, this infinite series converges precisely when $|z| < 1$.

An alternative solution is to apply Cauchy's root test. Viewed as $\sum_j a_j z^j$, the series has coefficients that are sometimes equal to 0 and sometimes equal to 1, and infinitely many coefficients are equal to 1. Therefore the largest limit point of the sequence $(|a_j|^{1/j})$ equals 1, and the power series has radius of convergence equal to the reciprocal of 1, which is still 1. Thus the series converges when $|z| < 1$ and diverges when $|z| > 1$. Resolving the case when $|z| = 1$ again requires observing that the terms do not tend to 0 when $|z| = 1$, so the series diverges when $|z| = 1$.

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4. Determine all values of the complex number z for which the infinite series

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$$

converges. (This series is *not* a power series, so the convergence region need not be a disk.) Justify your answer.

Solution. If $|z| < 1$, then $|z^{2n}| \le |z|$, so the triangle inequality implies that

$$\left|\frac{z^n}{1+z^{2n}}\right| \le \frac{|z|^n}{1-|z^{2n}|} \le \frac{|z|^n}{1-|z|}.$$

Therefore the infinite series converges absolutely by comparison with the convergent geometric series $\sum_{n=1}^{\infty} |z|^n$. Elementary algebra shows that the fraction

$$\frac{z^n}{1+z^{2n}}$$

is unchanged when z is replaced by $1/z$, so the infinite series converges also when $|z| > 1$. If $|z| = 1$, then the triangle inequality implies that

$$\left|\frac{z^n}{1+z^{2n}}\right| \ge \frac{1}{2}.$$

Since the terms of the series do not approach zero, the infinite series diverges when $|z| = 1$.

In summary, the infinite series converges if and only if $|z| \neq 1$.

5. State some version of each of the following theorems (with all hypotheses and conclusions correct).

   (a) Cauchy's theorem (about integrals being equal to zero)

   (b) Cauchy's integral formula

   (c) Morera's theorem

Solution. Theorems 2.4 and 2.5 and Corollary 2.8, for example.

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6. Explain why $\oint_C \left(z + \frac{1}{z}\right)^{102} dz$ equals 0 whenever C is a circle for which the integral makes sense (that is, the origin does not lie on the integration path).

Solution. If the origin is outside the circle, then the integral is equal to 0 by Cauchy's theorem, since the integrand is holomorphic everywhere inside the circle. If the origin is inside the circle, however, a different reason is needed.

Expand the integrand by the binomial formula to obtain a linear combination of terms of the form $z^k(1/z)^{102-k}$, equivalently z^{2k-102}, where k is an integer between 0 and 102. Since the exponent is even, each such term has an antiderivative along an arbitrary path that avoids the origin: namely, $z^{2k-101}/(2k-101)$. Since the path is closed, and the integrand is a derivative, the integral equals 0.

An alternative argument when the origin is inside the circle is to use the path-deformation principle to replace the path by the unit circle centered at the origin. Then the path can be parametrized by setting z equal to $e^{i\theta}$, and the integral becomes

$$\int_{-\pi}^{\pi} (2\cos\theta)^{102}\, i(\cos\theta + i\sin\theta)\, d\theta.$$

By symmetry,

$$\int_{-\pi}^{\pi} (\cos\theta)^{102} \sin\theta\, d\theta = 0.$$

Moreover, setting θ equal to $\frac{\pi}{2} - \psi$ and invoking periodicity and symmetry shows that

$$\int_{-\pi}^{\pi} (\cos\theta)^{103}\, d\theta = \int_{-\pi/2}^{3\pi/2} (\sin\psi)^{103}\, d\psi = \int_{-\pi}^{\pi} (\sin\psi)^{103}\, d\psi = 0.$$