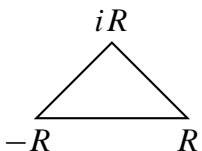# Theory of Functions of a Complex Variable I

Instructions   Please solve *six* of the following seven problems. Treat these problems as essay questions: supporting explanation is required.

1. When R is a real number greater than 1, let C_R denote the triangle (oriented counterclockwise) with vertices $-R$, R, and iR. Does the limit

$$\lim_{R \to \infty} \int_{C_R} \frac{1}{1 + z^2} \, dz \qquad \text{exist?}$$

Solution.   The singularities of the integrand are $\pm i$, but only i is inside the path C_R. By the path-deformation principle, the value of the integral does not change when the path is changed into a circle of radius 1 centered at i. Thus the integral is independent of R (when $R > 1$). Consequently, the limit of the integral certainly exists when $R \to \infty$. (The limit of a constant function exists!)

The problem does not ask for the actual value of the limit, but you know how to compute the value. The method of partial fractions shows that

$$\frac{1}{1 + z^2} = \frac{1}{2i} \left(\frac{1}{z - i} - \frac{1}{z + i} \right).$$

The integral of $1/(z + i)$ over C_R is equal to 0 by Cauchy's integral theorem, while the integral of $1/(z - i)$ over C_R is equal to $2\pi i$ by Cauchy's integral formula. Therefore the integral of $1/(1 + z^2)$ over C_R is equal to π.

An alternative approach to the problem is to apply the principle that the modulus of a complex line integral is at most the length of the path times the maximum of the modulus of the integrand on the path. The part of the path in the upper half-plane has length $2R\sqrt{2}$, and $|z| \geq R/\sqrt{2}$ on this part of the path. Therefore the modulus of the integral over the part of the path in the upper half-plane is bounded above by

$$\frac{2R\sqrt{2}}{\frac{1}{2}R^2 - 1} \qquad (\text{when } R > \sqrt{2}),$$

which tends to 0 when R tends to ∞. Accordingly, the limit of the integral over the whole of C_R equals

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 + x^2} \, dx,$$

or, from elementary calculus, $\lim_{R \to \infty} 2 \arctan(R)$. This limit evidently exists and equals π.

2. Let D denote the unit disk, $\{ z \in \mathbb{C} : |z| < 1 \}$. If $f : D \to D$ is a holomorphic function, then how big can $|f''(0)|$ be?

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Solution. The range of f is contained in the unit disk, so $\sup_{|z|<1}|f(z)| \leq 1$. Applying Cauchy's inequality for the second derivative on a disk of radius r slightly less than 1 shows that

$$\frac{|f''(0)|}{2!} \leq \frac{1}{r^2}.$$

The left-hand side is independent of r, so taking the limit as r tends to 1 shows that $|f''(0)| \leq 2$. This inequality is the best possible, for equality is realized by the squaring function ($z \mapsto z^2$).

(What is the point of shrinking the radius of the disk and then taking a limit? This technical device is needed because our proof of Cauchy's inequality uses that the function is defined on the *closed* disk.)

3. Riemann's famous zeta function can be defined as follows:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{when } \operatorname{Re} z > 1.$$

(Recall that n^z means $\exp\{z\ln(n)\}$ when n is a positive integer.) Notice that this infinite series is not a power series. Does this infinite series converge uniformly on each compact subset of the open half-plane $\{\, z \in \mathbb{C} : \operatorname{Re} z > 1 \,\}$?

Solution. Observe that $|n^z| = |\exp\{z\ln(n)\}| = \exp\{\operatorname{Re}(z)\ln(n)\} = n^{\operatorname{Re}(z)}$. Consequently, if p is a fixed positive real number, and $\operatorname{Re}(z) \geq p$, then

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^{\operatorname{Re}(z)}} \leq \frac{1}{n^p}.$$

When $p > 1$, the infinite series $\sum_{n=1}^{\infty} 1/n^p$ converges, so the Weierstrass M-test implies that the series for the zeta function converges uniformly in each closed half-plane where $\operatorname{Re}(z) \geq p > 1$. Each compact subset of the open half-plane $\{\, z \in \mathbb{C} : \operatorname{Re} z > 1 \,\}$ is contained in one of these closed half-planes, so the series for the zeta function does indeed converge uniformly on each compact subset of the open half-plane $\{\, z \in \mathbb{C} : \operatorname{Re} z > 1 \,\}$.

4. In what region of the complex plane does the integral

$$\int_0^1 \frac{1}{(1-zt)^2}\,dt$$

represent a holomorphic function of z? (The formula is to be understood as an integral in which the real variable t moves along the real axis from 0 to 1.)

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Solution. When z lies in the unbounded closed interval $[1, \infty)$ on the real axis, this integral diverges, for there is a non-integrable singularity on the integration path (where $t = 1/z$). There are various ways to check that the integral represents a holomorphic function on the complementary open set, $\mathbb{C} \setminus [1, \infty)$. Holomorphicity is a local property, so the checking can be done in a small open disk that has positive distance from the interval $[1, \infty)$. In such a disk, the modulus of the integrand is bounded uniformly with respect to t in the interval $[0, 1]$. Consequently, Morera's theorem can be applied to verify holomorphicity, for the uniform boundedness justifies interchanging the order of integration. Alternatively, almost any theorem you know about differentiating under the integral sign can be applied.

5. In what region of the complex plane does the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{z^n} + \frac{z^n}{2^n} \right)$$

represent a holomorphic function of z?

Solution. If r and R are real numbers such that $1 < r < R < 2$, and $r \leq |z| \leq R$, then

$$\left| \frac{1}{z^n} + \frac{z^n}{2^n} \right| \leq \left(\frac{1}{r} \right)^n + \left(\frac{R}{2} \right)^n.$$

Since $1/r < 1$ and $R/2 < 1$, the right-hand side is summable when n runs over the positive integers. The Weierstrass M-test implies that the original series converges uniformly when $r \leq |z| \leq R$. Letting r and R vary shows that the series converges uniformly on every compact subset of the open annulus $\{ z \in \mathbb{C} : 1 < |z| < 2 \}$ and so represents a holomorphic function on this annulus.

There is no larger set on which the series represents a holomorphic function. Indeed, there is no larger set on which the series converges. (If $|z| \geq 2$, then the first summand tends to 0 as n increases, but the second summand fails to tend to 0, so the infinite series diverges. If $|z| \leq 1$, then the second summand tends to 0, but the first summand fails to tend to 0, so the infinite series diverges.)

6. There cannot exist an entire function f with the property that

$$f\left(\frac{1}{n} \right) = \frac{(-1)^n}{n^2} \qquad \text{for every positive integer } n.$$

Why not?

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Solution. **Method 1** Suppose, seeking a contradiction, that such a function f does exist. By continuity, $f(0) = \lim\limits_{n\to\infty} f(1/n) = 0$. Moreover,

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{n\to\infty} \frac{f(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1)^n}{n},$$

so $f'(0) = 0$. Accordingly, the Maclaurin series of $f(z)$ has the form $c_2 z^2 + c_3 z^3 + \cdots$. Therefore

$$\lim_{z\to 0} \frac{f(z)}{z^2} \qquad \text{exists and equals } c_2.$$

But

$$\lim_{z\to 0} \frac{f(z)}{z^2} = \lim_{n\to\infty} \frac{f(1/n)}{1/n^2} = \lim_{n\to\infty} (-1)^n,$$

and the limit on the right-hand side does not exist. The contradiction shows that f cannot exist after all.

Method 2 If there were such a function f, then $f(z) - z^2$ would be equal to 0 when z is the reciprocal of an arbitrary even positive integer. By the coincidence principle, $f(z) \equiv z^2$. Similarly, $f(z) + z^2$ would be equal to 0 when z is the reciprocal of an arbitrary odd positive integer. By the coincidence principle, $f(z) \equiv -z^2$. Evidently $f(z)$ cannot be identically equal simultaneously to $+z^2$ and to $-z^2$. The contradiction shows that f cannot exist.

7. Prove the following property of the gamma function:

$$\left| \Gamma\left(\tfrac{1}{2} + it \right) \right|^2 = \frac{\pi}{\cosh(\pi t)} \qquad \text{when } t \in \mathbb{R}.$$

Hint: Recall that $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$, and $\cosh(z) = \tfrac{1}{2}(e^z + e^{-z})$.

Solution. The idea is simply to substitute $\tfrac{1}{2} + it$ for z in the functional equation stated in the hint.

A trigonometric identity says that $\sin(\tfrac{\pi}{2} + x) = \cos(x)$ when x is a real number. This identity persists when x is replaced by a complex number z (by the coincidence principle). Consequently,

$$\frac{\pi}{\sin\left(\pi \left(\tfrac{1}{2} + it \right) \right)} = \frac{\pi}{\cos(\pi i t)} = \frac{\pi}{\cosh(\pi t)}.$$

On the other hand,

$$\Gamma\left(\tfrac{1}{2} + it \right) \Gamma\left(1 - \left(\tfrac{1}{2} + it \right) \right) = \Gamma\left(\tfrac{1}{2} + it \right) \Gamma\left(\tfrac{1}{2} - it \right).$$

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Accordingly, all that remains to show is that

$$\Gamma\left(\tfrac{1}{2} - it\right) = \overline{\Gamma\left(\tfrac{1}{2} + it\right)}$$

when t is an arbitrary real number.

One way to justify this equality is to invoke the integral that defines the gamma function in the right-hand half-plane (changing the dummy variable of integration to s, since the letter t is already in use):

$$\Gamma\left(\tfrac{1}{2} - it\right) = \int_0^\infty s^{-\frac{1}{2}-it} e^{-s}\, \mathrm{d}s \qquad \text{and} \qquad \Gamma\left(\tfrac{1}{2} + it\right) = \int_0^\infty s^{-\frac{1}{2}+it} e^{-s}\, \mathrm{d}s.$$

Since the complex conjugate of an integral is the integral of the complex conjugate, and the complex conjugate of $s^{-\frac{1}{2}-it}$ is $s^{-\frac{1}{2}+it}$, the desired result follows. An alternative method is to observe that the gamma function takes real values on the positive part of the real axis, so $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ for every z by a corollary of the coincidence principle (see page 56 of the textbook).