Examination 2

1. Suppose f is a holomorphic function in the right-hand half-plane such that $f'(z) = 1/z$ when $\operatorname{Re}(z) > 0$, and $f(1) = i$. Find the value of $f(1+i)$ in the form $a + bi$.

   Solution. One function whose derivative equals $1/z$ is the principal branch of the logarithm. Two functions having the same derivative (on a connected open set) differ by a constant, so there is some complex number c such that $f(z) = c + \log(z)$. Since $f(1) = i$, and $\log(1) = 0$, the constant c is equal to i. Then $f(1+i) = i + \log(1+i) = i + \ln\left(\sqrt{2}\right) + \frac{\pi}{4}i = \frac{1}{2}\ln 2 + (1 + \frac{\pi}{4})i$.

2. Suppose f has an isolated singularity at 0, and the residue of f at 0 is equal to 4. Suppose $g(z) = f(2z) + 3f(z)$ for all z in a punctured neighborhood of 0. Find the residue of g at 0.

   Solution.

   Method 1 By hypothesis, there are coefficient sequences (a_n) and (b_n) such that the Laurent series for f has the following form:

   $$\sum_{n=-\infty}^{-2} a_n z^n + \frac{4}{z} + \sum_{n=0}^{\infty} b_n z^n.$$

   The transformation $z \mapsto 2z$ (in the domain) and the transformation $w \mapsto 3w$ (in the range) map each monomial z^n to a multiple of the same monomial, so there are coefficient sequences $(\widetilde{a}_n)$ and $(\widetilde{b}_n)$ such that the Laurent series of g has the form

   $$\sum_{n=-\infty}^{-2} \widetilde{a}_n z^n + \frac{4}{2z} + 3 \cdot \frac{4}{z} + \sum_{n=0}^{\infty} \widetilde{b}_n z^n.$$

   [Explicit values for the coefficients are easy to find: namely, $\widetilde{a}_n = (2^n + 3)a_n$, and $\widetilde{b}_n = (2^n + 3)b_n$.] Therefore the residue of g at 0 equals $\frac{4}{2} + 3 \cdot 4$, or 14.

   Method 2 There is a positive radius r such that f is holomorphic at least in the punctured disk of radius $3r$ centered at 0. The residue of f at 0 equals $(2\pi i)^{-1}$ times the integral of f around an arbitrary counterclockwise circle centered at 0 of radius less than $3r$, so

   $$\operatorname{Res}(f, 0) = \frac{1}{2\pi i} \oint_{|z|=r} f(z)\, dz = \frac{1}{2\pi i} \oint_{|z|=2r} f(z)\, dz.$$

Similarly,

$$\begin{aligned}\operatorname{Res}(g,0) &= \frac{1}{2\pi i}\oint_{|z|=r} g(z)\,dz = \frac{1}{2\pi i}\oint_{|z|=r} (f(2z)+3f(z))\,dz\\ &= \frac{1}{2\pi i}\oint_{|z|=r} f(2z)\,dz + 3\operatorname{Res}(f,0).\end{aligned}$$

The remaining integral can be computed by replacing $2z$ with a new variable w as follows:

$$\frac{1}{2\pi i}\oint_{|z|=r} f(2z)\,dz = \frac{1}{2\pi i}\oint_{|w|=2r} f(w)\,\frac{dw}{2} = \frac{1}{2}\operatorname{Res}(f,0).$$

The conclusion is that

$$\operatorname{Res}(g,0) = \left(\frac{1}{2}+3\right)\operatorname{Res}(f,0) = 14.$$

3. Suppose $\gamma_0 \colon [0,1] \to \mathbb{C}\setminus\{0\}$ is a differentiable closed curve lying in the punctured plane, and $\gamma_1(t) = \gamma_0(t^2)$ when $0 \le t \le 1$. If the index (winding number) of γ_0 about the origin is equal to 5, what is the value of the index of γ_1 about the origin? Explain how you know.

Solution. The function $t \mapsto t^2$ is a homeomorphism (that is, a bicontinuous bijection) from the interval $[0,1]$ onto $[0,1]$. Accordingly, the functions γ_0 and γ_1 are simply different parametrizations of the same geometric curve. Therefore γ_0 and γ_1 had better have the same winding number about the origin. There are multiple ways to confirm this intuition.

Method 1 The winding number can be expressed as an integral as follows:

$$\begin{aligned}\operatorname{Ind}(\gamma_0,0) &= \frac{1}{2\pi i}\int_{\gamma_0}\frac{1}{z}\,dz = \frac{1}{2\pi i}\int_0^1 \frac{1}{\gamma_0(t)}\gamma_0'(t)\,dt\\ &\overset{t=s^2}{=} \frac{1}{2\pi i}\int_0^1 \frac{1}{\gamma_0(s^2)}\gamma_0'(s^2)\,2s\,ds = \frac{1}{2\pi i}\int_0^1 \frac{1}{\gamma_1(s)}\gamma_1'(s)\,ds\\ &= \frac{1}{2\pi i}\int_{\gamma_1}\frac{1}{z}\,dz = \operatorname{Ind}(\gamma_1,0).\end{aligned}$$

Method 2 Define a function Γ of two variables as follows:

$$\Gamma(s,t) = \gamma_0(st + (1-s)t^2).$$

If $0 \le s \le 1$ and $0 \le t \le 1$, then $0 \le st + (1-s)t^2 \le s + (1-s) = 1$, so Γ is well defined on $[0,1] \times [0,1]$. Obtained by composing continuous functions, the function Γ is continuous. Evidently $\Gamma(0,t) = \gamma_0(t^2) = \gamma_1(t)$, and $\Gamma(1,t) = \gamma_0(t)$. Moreover $\Gamma(s,0) = \gamma_0(0) = \gamma_0(1) = \Gamma(s,1)$, so for each fixed value of s, the function sending t to $\Gamma(s,t)$ represents a closed curve. The range of Γ is identical to the range of γ_0.

In summary, the function Γ is a homotopy between the two closed curves γ_1 and γ_0 in the region $\mathbb{C} \setminus \{0\}$. The winding number is a homotopy invariant, so γ_0 and γ_1 have the same winding number about the origin.

Method 3 By Lemma 4.6 on page 57 of the textbook, there is a continuous function θ such that $\gamma_0(t) = |\gamma_0(t)|e^{i\theta(t)}$, and $\operatorname{Ind}(\gamma_0, 0) = \frac{1}{2\pi}(\theta(1) - \theta(0))$. Replace t by t^2 to deduce that $\gamma_1(t) = |\gamma_1(t)|e^{i\theta(t^2)}$. Accordingly,

$$\operatorname{Ind}(\gamma_1, 0) = \frac{1}{2\pi}\left(\theta(t^2)\,\Big|_{t=1} - \theta(t^2)\,\Big|_{t=0}\right) = \frac{1}{2\pi}(\theta(1) - \theta(0)) = \operatorname{Ind}(\gamma_0, 0).$$

4. Find the maximum value of $|i + z^2|$ when $|z| \le 2$.

Solution.

Method 1 By the triangle inequality, $|i + z^2| \le |i| + |z^2| \le 1 + 2^2$ when $|z| \le 2$. On the other hand, this upper bound 5 is attained when $z = 2e^{i\pi/4}$. Therefore the maximum value is 5.

Method 2 The function sending z to z^2 maps the disk of radius 2 onto the disk of radius 4. Adding i translates this disk of radius 4 one unit upward in the plane. The point in the translated disk at greatest distance from the origin evidently is the highest point on the imaginary axis, the point $5i$. Therefore the maximum value of $|i + z^2|$ is 5.

Method 3 By the maximum principle, the maximum occurs on the boundary of the disk. Therefore the problem reduces to maximizing $|i + 4e^{2i\theta}|$ when θ varies from 0 to 2π. In other words, the goal is to maximize

$$\sqrt{(4\cos(2\theta))^2 + (1 + 4\sin(2\theta))^2},$$

which simplifies to $\sqrt{17 + 8\sin(2\theta)}$. Evidently the maximum occurs when $\sin(2\theta) = 1$, and the value of the maximum is $\sqrt{17+8}$, or 5.

5. Prove that $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}$.
(This integral can—in principle—be evaluated by using techniques of real calculus, but you are more likely to be successful by applying the residue theorem.)

Solution.

Method 1 Integrate $\frac{1}{1+z^4}$ over the contour in the (closed) first quadrant that starts at 0, travels along the real axis to a value R greater than 1, follows the circle of radius R to the point Ri, and travels down the imaginary axis back to 0. By the residue theorem, this integral equals $2\pi i$ times the residue of $\frac{1}{1+z^4}$ at the simple pole where $z = e^{\pi i/4}$. In other words, this integral equals

$$2\pi i \cdot \frac{1}{4e^{3\pi i/4}}, \qquad \text{or} \qquad \frac{\pi}{2e^{\pi i/4}}.$$

The part of the integral along the real axis equals $\int_0^R \frac{1}{1+x^4}\,dx$, and the part of the integral along the imaginary axis equals $\int_R^0 \frac{1}{1+y^4}\,i\,dy$, or $-i\int_0^R \frac{1}{1+y^4}\,dy$. On the circular arc, where $z = Re^{i\theta}$, the modulus of the integrand is at most $\frac{1}{R^4-1}$. Since the length of the circular arc is $\pi R/2$, the integral over the circular arc is $O(1/R^3)$ as $R \to \infty$.

In summary, the residue theorem implies that

$$\frac{\pi}{2e^{\pi i/4}} = (1-i)\int_0^R \frac{1}{1+x^4}\,dx + O(1/R^3).$$

Taking the limit as $R \to \infty$ and dividing by $1 - i$ shows that

$$\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi}{2e^{\pi i/4}(1-i)} = \frac{\pi}{2\sqrt{2}},$$

as required.

Method 2 By symmetry, $\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1+x^4}\,dx$. Consider the integral of $\frac{1/2}{1+z^4}$ over the contour that travels along the real axis from $-R$ to R

(where $R > 1$) and then follows the circle of radius R in the upper half-plane from R to $-R$. For the same reason as in Method 1, the integral over the semi-circle is $O(1/R^3)$. Therefore the limit as $R \to \infty$ of the integral of $\frac{1/2}{1+z^4}$ over the closed contour equals the value of the integral stated in the problem.

By the residue theorem, this value equals $2\pi i$ times the sum of the residues of $\frac{1/2}{1+z^4}$ at the two (simple) poles in the upper half-plane, where $z = e^{\pi i/4}$ and $z = e^{3\pi i/4}$. Accordingly, the value of $\int_0^\infty \frac{1}{1+x^4}\,dx$ equals

$$2\pi i \left(\frac{1/2}{4(e^{\pi i/4})^3} + \frac{1/2}{4(e^{3\pi i/4})^3} \right).$$

This expression simplifies as follows:

$$\begin{aligned} \frac{\pi i}{4}\left(\frac{1}{e^{3\pi i/4}} + \frac{1}{e^{9\pi i/4}}\right) &= \frac{\pi i}{4}\left(e^{-3\pi i/4} + e^{-\pi i/4}\right) \\ &= \frac{\pi}{4}\left(e^{-\pi i/4} + e^{\pi i/4}\right) \\ &= \frac{\pi}{4}\left(2\cos(\pi/4)\right) \\ &= \frac{\pi}{2\sqrt{2}}. \end{aligned}$$

6. Let V denote an open subset of $\mathbb{C}$, let γ denote a differentiable simple closed curve that lies in V, and let f denote a holomorphic function in V. Answer any two of the following three questions.

   (a) What additional property of γ is necessary and sufficient to guarantee that $\int_\gamma f(z)\,dz = 0$ for every f?

   Solution. A necessary and sufficient condition is that $\operatorname{Ind}(\gamma, w) = 0$ for every point w in the complement of V. See Theorem 4.10 (the homology version of Cauchy's theorem) and the subsequent discussion on page 62 of the textbook.

   Since γ is simple, the curve γ bounds a Jordan region, whence γ is homologous to zero if and only if γ is homotopic to a constant curve. Therefore you could also say—under the special hypotheses in force in this problem—that a necessary and sufficient condition is for γ to be homotopic in V to a constant curve.

(b) What additional property of f is necessary and sufficient to guarantee that $\int_\gamma f(z)\,dz = 0$ for every γ?

Solution. This question is answered by Theorem 4.0 on page 52 of the textbook: the necessary and sufficient condition is the existence of a holomorphic anti-derivative of f in V.

(c) What additional property of V is necessary and sufficient to guarantee that $\int_\gamma f(z)\,dz = 0$ for every f and every γ?

Solution. If V is simply connected, then $\int_\gamma f(z)\,dz = 0$ for every f and every γ by Theorem 4.14. The validity of the converse is not made explicit in the textbook until Chapter 11, but the reason is easy to state intuitively. If V fails to be simply connected, then there is a hole in V. Draw a simple closed counterclockwise curve γ around the hole, pick a point w in the hole, and observe that $\int_\gamma \frac{1}{z-w}\,dz = 2\pi i \neq 0$.