The zeta function

The goal of this exercise is to understand the definition of the zeta function.

You have likely seen formulas like $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. What happens if the power of n is replaced by some other number? The result is Riemann's zeta function:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \operatorname{Re} z > 1. \tag{1}$$

1. Since the complex power $a^b := e^{b \log a}$ is "multi-valued", is the definition of $\zeta(z)$ ambiguous?

2. Why does the definition (1) produce a *holomorphic* function?

It turns out that the ζ function can be continued analytically to $\mathbb{C} \setminus \{1\}$, with a simple pole at 1. Our goal is to prove a little less.

3. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z} = (2^{1-z} - 1)\zeta(z)$ when $\operatorname{Re} z > 1$ by grouping the terms in the absolutely convergent sum on the left-hand side according to the parity of n.

4. Assuming that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$ represents a holomorphic function when $\operatorname{Re} z > 0$, deduce that ζ is a meromorphic function when $\operatorname{Re} z > 0$ and that ζ has a simple pole at 1 with residue 1.

5. A generalization of the alternating series test states that sufficient conditions for convergence of a series $\sum_{n=1}^{\infty} (-1)^n b_n$ are that the sequence $\{b_n\}$ has bounded variation, meaning $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$, and that $b_n \to 0$. (An analogous statement holds for uniform convergence.) To complete the preceding argument, deduce from this convergence test that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$ does represent a holomorphic function when $\operatorname{Re} z > 0$.