Exercise on Jensen's formula

The goal of this exercise is to demonstrate that you understand and can apply Jensen's formula.

Recall *Jensen's formula* (Theorem 9.1.2 on page 281 of the textbook) says that if f is analytic in a neighborhood of the closed disk $\overline{D}(0,r)$, if $f(0) \neq 0$ (this is just a convenient normalization), and if the zeroes of f in $D(0,r)$ are a_1, a_2, $\ldots$, a_n (repeated according to multiplicity), then

$$\log|f(0)| + \sum_{j=1}^{n} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta.$$

If f is zero-free, then there is no sum on the left-hand side, and the formula is just the mean-value property for the harmonic function $\log|f|$. The general case can be reduced to this special one by dividing f by a suitable Blaschke product to cancel its zeroes.

The importance of Jensen's formula is that it gives a connection between the number of zeroes of an entire function and its rate of growth.

1. Suppose f is an entire function, and let $n(t)$ denote the number of zeroes of f (counted with multiplicity) having modulus less than t. Reinterpret Jensen's formula as saying that

$$\log|f(0)| + \int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta.$$

2. Let $M(r)$ denote the maximum modulus of the entire function f on the circle $\{\, z : |z| = r \,\}$. Deduce that if f is normalized via $f(0) = 1$, then

$$n(r)\log 2 \leq \log M(2r).$$

This inequality is a quantitative expression of the principle that an entire function with many zeroes must grow rapidly.

The inequality is given in the textbook as Lemma 9.3.1, and an alternate proof of the inequality is presented on page 290.