Exercise on the definition of order

The goal of this exercise is to demonstrate that you understand the definition of the order of an entire function.

Let $M(r)$ denote the maximum of the modulus of an entire function f on the disk of radius r centered at the origin; that is, $M(r) := \max_{|z|\le r} |f(z)|$. The growth rate of $M(r)$ as $r \to \infty$ tells a lot about the function f. For example, if $\lim_{r\to\infty} \frac{M(r)}{r^k} = 0$, then f must be a polynomial of degree less than k (by a generalization of Liouville's theorem: Theorem 3.4.4, page 89).

The word *order* is used to describe the exponential growth rate of an entire function. Here are four examples: the order of the exponential function e^z is 1; the order of e^{z^2} is 2; the order of a polynomial is 0; the order of z^3e^z is 1. Usually one denotes order by the letter λ. A precise definition is

$$\lambda := \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

1. Verify that the formal definition gives the right value for the four examples.

2. Verify that the order λ is the smallest real number satisfying the following property: for every positive ϵ, there exists a positive R such that $M(r) \le e^{r^{\lambda+\epsilon}}$ when $r \ge R$.

3. Verify that the order λ is the smallest real number satisfying the following property: for every positive ϵ, there exists a positive C such that $M(r) \le Ce^{r^{\lambda+\epsilon}}$ for *all* r.

4. What can you say about the order of

- the sum of two entire functions?
- the product of two entire functions?