Introduction to analytic continuation

Here are three related questions concerning a holomorphic function f defined on some connected open set (the unit disk, for example):

- Can f be extended to a holomorphic function on a larger domain?
- If so, is the extension unique?
- Is there a largest domain to which f can be extended holomorphically?

These questions form the subject of *analytic continuation*.

The answers to all three questions are "sometimes". This exercise discusses just the first question.

1. The series $\sum_{n=1}^{\infty} z^{n!}$ converges in the open unit disk, and the function defined by the series cannot be extended holomorphically to a neighborhood of any point of the unit circle. Why?

2. The series $\sum_{n=1}^{\infty} (-1)^n z^{2n}$ converges in the open unit disk, and in no larger open disk. The function defined by the series does extend holomorphically to a neighborhood of some points of the unit circle. Which ones?

3. The inverse of the Cayley transform maps the unit disk biholomorphically onto the upper half-plane. This function on the unit disk extends holomorphically to a neighborhood of every point of the unit circle except for one. Show this in two different ways:

- by an explicit formula;
- by the Schwarz reflection principle.