Exercise on Riemann surfaces

The goal of this exercise is to demonstrate that you understand the basic notions of Riemann surfaces.

You have seen that there is no largest region of the plane $\mathbb{C}$ to which the function $\sqrt{z}$ continues analytically (largest in the sense of containing every other such region). There is, however, a natural maximal domain of definition for $\sqrt{z}$ which is a two-sheeted *Riemann surface* lying over the complex plane.

If you are familiar with the notion of a manifold, then you will recognize a Riemann surface as being a one-dimensional complex manifold. What this means is the following.

a. A Riemann surface is first of all a *Hausdorff topological space*. (It is equipped with a collection of open sets, and distinct points can be surrounded by disjoint open sets.)

b. There is a special collection (finite or infinite) of open sets, called *coordinate charts*, each of which is homeomorphic to an open subset of $\mathbb{C}$. Every point of the Riemann surface lies in at least one coordinate chart.

c. Whenever two charts overlap, composing one homeomorphism with the inverse of the other one induces a homeomorphism between certain open subsets of $\mathbb{C}$. Each such *transition function* should be biholomorphic.

1. A trivial example of a Riemann surface is an open subset of $\mathbb{C}$. Why?

A more interesting example of a Riemann surface is the extended complex numbers $\mathbb{C} \cup \{\infty\}$ modeled as the "Riemann sphere" $\{\,(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\,\}$ via stereographic projection $z = (x_1 + ix_2)/(1 - x_3)$. (See section 4.7 and exercises 32–34 on page 151.)

The sphere, being a metric space, is certainly a Hausdorff topological space. Two coordinate charts suffice: one chart is the sphere minus the north pole, which maps homeomorphically onto $\mathbb{C}$ via stereographic projection; the other chart is the sphere minus the south pole, which maps homeomorphically onto $\mathbb{C}$ by composing the mapping $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$ with stereographic projection.

2. On the overlap of this pair of coordinate charts, what holomorphic transition function is induced on what subset of $\mathbb{C}$?