

## Introduction to Mittag-Leffler's theorem

The goal of this exercise is to understand the main idea of Mittag-Leffler's theorem through a specific example.

The theorem of Weierstrass guarantees the existence of a holomorphic function with prescribed zeroes (under the necessary condition that the zeroes have no accumulation point inside the domain of the function). Mittag-Leffler's theorem addresses the problem of finding a function with prescribed *poles*. Let's consider the concrete example of poles at the positive integers.

1. Deduce from the theorem of Weierstrass that there exists a function meromorphic in the whole plane, with a simple pole at each positive integer  $n$ , and with no other poles.

Can you write down such a meromorphic function explicitly?

It is useful to know that both the poles and the *residues* of a meromorphic function can be prescribed. For example, consider the problem of constructing a meromorphic function having at each positive integer  $n$  a simple pole with residue equal to  $n$  (and no other poles).

It is hard to see how to modify an infinite product to adjust the residues. A more natural approach is to work with a generalized "partial fractions" decomposition.

2. A first try is to use the series  $\sum_{n=1}^{\infty} \frac{n}{z-n}$  to define the required meromorphic function. What goes wrong?

The remedy for the deficiency in this natural first guess is analogous to the idea in the proof of the Weierstrass theorem, except that one uses additive factors instead of multiplicative factors.

3. Show that there exist polynomials  $p_n$  such that the infinite series

$$\sum_{n=1}^{\infty} \left( \frac{n}{z-n} - p_n(z) \right)$$

converges uniformly on compact subsets of  $\mathbb{C} \setminus \{1, 2, \dots\}$  to a meromorphic function with the required properties.

What is the simplest choice for  $p_n$  that works?