Exercise on the proof of Runge's theorem

The goal of this exercise is to demonstrate understanding of the ideas in the proof of Runge's theorem by considering a specific example.

Recall the statement of the simplest version of Runge's theorem: if K is a compact set in $\mathbb{C}$ with connected complement, and if f is holomorphic in an open neighborhood of K, then f can be approximated uniformly on K by holomorphic polynomials. The proof is based on the following scheme: write Cauchy's integral formula, approximate the integral by Riemann sums, and then "push the poles" off to infinity.

As a specific example, let's take K to be the closed unit square in the complex plane. We want to show that if f is holomorphic in an open neighborhood of K, and if a positive ϵ is prescribed, then there exists a holomorphic polynomial p such that $\max_{z\in K} |f(z) - p(z)| < \epsilon$.

1. There is a slightly larger square such that by Cauchy's formula, we can express $f(z)$ for $z \in K$ as an integral over the boundary of the larger square.
2. By approximating the integral by a Riemann sum, we can approximate $f(z)$ for $z \in K$ by a finite linear combination of functions of the form $\dfrac{1}{(a_n - z)}$, where the a_n are points outside K. You need to verify that the approximation depends on z in a uniform way.

At this point, we have approximated f by a *rational* function with poles outside K (but close to K). We need to push the poles far away from K. The trick for doing this is to write $\dfrac{1}{z-a} = \dfrac{1}{z-b} \cdot \dfrac{1}{1 + \frac{b-a}{z-b}}$ and to truncate a convergent geometric series expansion.

3. When $a \notin K$, the function $\dfrac{1}{(z-a)}$ can be approximated uniformly on K by a linear combination of powers of $\dfrac{1}{(z-b)}$, where b is farther away from K than a is.
4. If b is far enough away from K, then $\dfrac{1}{(z-b)}$ can be replaced by a partial sum of its Maclaurin series to finish the construction of a polynomial approximation of f on K.