Exercise on Alice Roth's Swiss cheese

The goal of this exercise is to understand the limits of Mergelyan's theorem by explicating a counterexample.

Here is the question: if K is a compact set in $\mathbb{C}$, and f is continuous on K and holomorphic in the interior of K, can f be approximated uniformly on K by rational functions?

Mergelyan's theorem says that the answer is "yes" when the complement of K is connected. Runge's theorem says that the answer is "yes" when f is holomorphic in an open neighborhood of K. In general, however, the answer is "not necessarily".

The Swiss mathematician Alice Roth constructed[1] a counterexample K by punching an infinite number of holes in a planar domain. An example of this type is naturally called a "Swiss cheese", since this is the generic English name for a pale-yellow cheese (such as Emmenthaler or Gruyère) having many holes.

1. Show how to construct an infinite sequence $\{D_j\}_{j=1}^{\infty}$ of open disks satisfying the following properties.

(a) The closures of the disks D_j are pairwise disjoint and contained in the open unit disk D.

(b) The sum of the radii of the disks D_j is less than 1/2.

(c) The compact set $K := \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ has empty interior.

2. Prove that if f is a continuous function on K that can be approximated uniformly on K by rational functions, then

$$\int_{\partial D} f(z)\,dz = \sum_{j=1}^{\infty} \int_{\partial D_j} f(z)\,dz.$$

3. Deduce that although the function f defined by $f(z) = \bar{z}$ is continuous on K and holomorphic in the interior of K (vacuously, since K has empty interior), this function cannot be approximated uniformly on K by rational functions.

[1] Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen, *Commentarii Mathematici Helvetici* **11** (1938) 77–125.