Exercise on Picard's theorems

The goal of this exercise is to understand a second proof of Picard's theorem, a proof that does not use the modular function. This proof, different from the one in the textbook, has the advantage that it also yields Picard's great theorem. On the other hand, the proof is not completely self-contained: it requires knowledge of a deep result.

The basis of this proof is a fundamental theorem of Montel: *The family of holomorphic functions on an open set in the plane whose range omits the values* 0 *and* 1 *is a normal family.* (Here normality is understood in the generalized sense that the constant ∞ is an allowed limit function.)

1. The values 0 and 1 are just a convenient normalization: any two distinct complex numbers a and b would work as well. Why?

Let's assume Montel's theorem and use it to prove Picard's theorems.

Theorem (Picard's little theorem). *The range of a nonconstant entire function cannot omit two values.*

2. To prove this, suppose that f is an entire function whose range does omit two values, and consider the family of entire functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(z) = f(nz)$.

Theorem (Picard's great (big) theorem). *In every neighborhood of an essential singularity, a holomorphic function assumes every complex value infinitely often, with one possible exception.*

In other words, if f is holomorphic in a punctured disk, and if the isolated singularity at the center of the disk is an essential singularity, then the range of f omits at most one value; and no matter how small we shrink the punctured disk, the range of the restriction of f to the smaller disk still omits at most one value.

3. To prove this, assume without loss of generality that the puncture is at the origin, suppose—seeking a contradiction—that the range of f does omit two values, and consider the family of functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(z) = f(z/n)$.

The second theorem subsumes the first one, because a nonconstant entire function has at infinity either an essential singularity or a pole; in the latter case the function is a polynomial.