Exercise on Alice Roth's Swiss cheese

If K is a compact set in $\mathbb{C}$, and f is continuous on K and holomorphic in the interior of K, can f be approximated uniformly on K by rational functions?

S. N. Mergelyan (С. Н. Мергелян) proved that the answer is "yes" when the diameters of the components of the complement of K are bounded away from 0. Runge's theorem says that the answer is "yes" when f has a holomorphic extension to an open neighborhood of K. In general, however, the answer is sometimes "no".

The Swiss mathematician Alice Roth constructed a counterexample by punching an infinite number of holes in a disc.[1] Such a set is now called a "Swiss cheese" (which is the generic name in English for a pale-yellow cheese having many holes). Your task is to recreate a version of Alice Roth's example, as follows. (See also starred exercise 18 on page 381 of the textbook.)

1. Construct a sequence $\{D_j\}_{j=1}^{\infty}$ of open discs satisfying all three of the following properties.

(a) The closures of the discs D_j are pairwise disjoint and are contained in the open unit disc D.

(b) The sum of the radii of the discs D_j is less than 1/2.

(c) The set $K := \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ is compact and has empty interior.

2. Use Cauchy's theorem to show that if R is a rational function whose poles lie in the complement of K, then

$$\int_{\partial D} R(z)\,dz = \sum_{j=1}^{\infty} \int_{\partial D_j} R(z)\,dz.$$

3. Observe that $\displaystyle\int_{\partial D} \bar{z}\,dz = 2\pi i$, while $\displaystyle\left|\sum_{j=1}^{\infty} \int_{\partial D_j} \bar{z}\,dz\right| < \pi.$

4. Deduce that the function f defined by $f(z) = \bar{z}$ is an example of a continuous function on K that is holomorphic in the interior of K (vacuously, since K has empty interior) and that cannot be arbitrarily well approximated uniformly on K by rational functions.

[1] Alice Roth, Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen, *Commentarii Mathematici Helvetici* **11** (1938) 77–125.