Variations on the theorem of Weierstrass and Mittag-Leffler

This exercise aims to unify the Weierstrass and Mittag-Leffler theorems.

1. Deduce from the Weierstrass theorem the following weak unified statement. If U is an open set in the complex plane, $\{a_n\}$ is a discrete set in U, and $\{k_n\}$ is a sequence of integers, then there exists a meromorphic function in U with no zeroes or poles outside of $\{a_n\}$ and such that for each n, the Laurent series of the function at a_n starts with the power $(z - a_n)^{k_n}$.

Stimulated by Weierstrass's lectures in 1875, Mittag-Leffler worked out an improved theorem and eventually published it in the international journal that he founded.[1] Suppose given two sequences of polynomials, $\{p_n\}$ and $\{q_n\}$. Then there exists a meromorphic function f on U with no zeroes or poles outside of $\{a_n\}$ and such that for each n, the Laurent series of f at a_n is $p_n(1/(z - a_n)) + q_n(z - a_n)$ plus higher-order terms. The proof can be executed in two steps, as follows.

2. Use the Weierstrass theorem to find a holomorphic function g having for each n a zero at a_n of order $1 + \deg q_n$. Then use the version of Mittag-Leffler's theorem that you already know to find a meromorphic function h having for each n the same principal part at a_n as the quotient $[p_n(1/(z - a_n)) + q_n(z - a_n)]/g(z)$. Show that the product $f_1 := gh$ has for each n a Laurent series at a_n of the desired form. (This argument is a short proof of Theorem 8.3.8 in the textbook.)

The function f_1 just determined is almost the required f. The second step in the proof is to remove extraneous zeroes from f_1.

3. By part 1, there is a meromorphic function f_2 having zeroes and poles of the same orders as f_1 at the $\{a_n\}$ and having no other zeroes or poles. (The Laurent series coefficients of f_2 are not under control, however.) Locally near each point a_n, one can define a holomorphic branch of $\log(f_1/f_2)$, and by part 2 there is a holomorphic function φ on U that agrees to suitably high order at each a_n with $\log(f_1/f_2)$. Set $f := f_2 e^{\varphi}$.

[1] G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," *Acta Math.* **4** (1884), 1–79.