Montel's theorem

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Paul Montel, 1876–1975

He wrote *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Paris, 1927.

Montel's fundamental normality criterion

Theorem. *Let G be a domain in $\mathbb{C}$, let α and β be two distinct complex numbers, and let $\mathcal{F}$ be the set of holomorphic functions in G whose range omits the two values α and β. Then $\mathcal{F}$ is a normal family in the extended sense (that is, the constant ∞ is an allowed limit).*

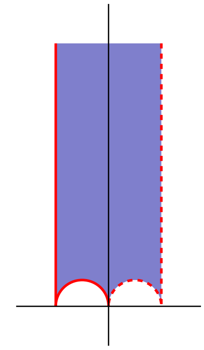
Reductions:

- We may assume that the two omitted values are 0 and 1. (Consider $(f(z) - \alpha)/(\beta - \alpha)$.)
- We may assume that the domain G is the unit disc. (Normality is a local property.)
- It suffices to prove normality of the smaller family $\mathcal{F}_1$ such that $\mathcal{F}_1 = \{ f \in \mathcal{F} : |f(0)| \leq 1 \}$. (If $f \notin \mathcal{F}_1$, then $1/f \in \mathcal{F}_1$.)

Reminders on the modular function λ

The modular function λ maps the open upper half-plane onto $\mathbb{C} \setminus \{0, 1\}$.

The modular function λ is invariant under the action of the congruence subgroup of the modular group. That is, $\lambda\left(\dfrac{az+b}{cz+d}\right) = \lambda(z)$ when a, b, c, and d are integers such that $ad - bc = 1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2$.



fundamental domain

Apply the monodromy theorem

If f maps the unit disc into $\mathbb{C} \setminus \{0, 1\}$, then $\lambda^{-1} \circ f$ has a local branch defined near $f(0)$, and this function element admits unrestricted analytic continuation in the disc.

By the monodromy theorem, there exists a function $\widehat{f}$ from the unit disc into the upper half-plane such that $\lambda \circ \widehat{f} = f$.

First subsequence

Suppose $\{ f_n : n \in \mathbb{N} \}$ is a sequence of functions in the family $\mathcal{F}_1$. We need to produce a normally convergent subsequence.

Since the numbers $f_n(0)$ lie in a bounded set, there is a subsequence $\{ n(k) : k \in \mathbb{N} \}$ such that the numbers $f_{n(k)}(0)$ converge to some complex number L.

Suppose first that $L \neq 0$ and $L \neq 1$. (We will return to the special cases later.) We fix a branch of λ^{-1} in a neighborhood of L and use it to define the functions $\widehat{f_{n(k)}}$ consistently.

Second subsequence

Since each $\widehat{f_{n(k)}}$ has range contained in the upper half-plane, the sequence $\{ \widehat{f_{n(k)}} : k \in \mathbb{N} \}$ is a normal family. Let $\{ n(k(j)) : j \in \mathbb{N} \}$ be a subsequence such that the functions $\widehat{f_{n(k(j))}}$ converge normally to a limit function g.

The range of the limit function g is certainly contained in the *closed* upper half-plane. Since $g(0) = \lambda^{-1}(L)$, the open mapping principle implies that the range of g is contained in the *open* upper half-plane.

Consequently, $\lambda \circ g$ is defined, and

$$f_{n(k(j))} = \lambda \circ \widehat{f_{n(k(j))}} \xrightarrow{j \to \infty} \lambda \circ g.$$

We are done in the main case.

Handle the remaining cases

Suppose $f_{n(k)}(0) \xrightarrow{k\to\infty} 1$. Let h_k be a holomorphic square-root of the non-vanishing function $f_{n(k)}$ with the branch chosen such that $h_k(0) \xrightarrow{k\to\infty} -1$. Clearly the range of each function h_k omits the values 0 and 1.

The preceding analysis applies to the sequence $\{ h_k : k \in \mathbb{N} \}$ and shows that there is a normally convergent subsequence $\{ h_{k(j)} : j \in \mathbb{N} \}$. Squaring shows that the sequence $\{ f_{n(k(j))} : j \in \mathbb{N} \}$ converges.

Finally, suppose that $f_{n(k)}(0) \xrightarrow{k\to\infty} 0$. The preceding case applies to the functions $1 - f_{n(k)}$.

We are done.