Exercises for discussion

The goal of these exercises is to solidify your understanding of the modular function: its construction and its application to Picard's little theorem.

1. The construction of the modular function starts with the set $\{\, z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \text{ and } 0 < \operatorname{Im} z \text{ and } |z - \frac{1}{2}| > \frac{1}{2} \,\}$, a half-strip with a half-disc removed. Consider starting instead with the full half-strip $\{\, z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \text{ and } 0 < \operatorname{Im} z \,\}$. This half-strip can be mapped biholomorphically to the upper half-plane by a function f that extends continuously to the boundary and takes $\{0, 1, \infty\}$ to $\{0, 1, \infty\}$. (Why?)

Show by repeated use of the Schwarz reflection principle that the function f can be analytically continued to an entire function that takes real values on the real axis and is periodic with period 2.

(Can you identify the function f concretely?)

2. The principal branch of the logarithm function maps the upper half-plane biholomorphically to the strip $\{\, z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi \,\}$. The boundaries of both the domain and the image consist of straight lines, so the Schwarz reflection principle should apply. Why doesn't this produce an analytic continuation of the logarithm function to the whole plane?

What does the Schwarz reflection principle produce in this case?

3. The proof of Picard's little theorem in the textbook depends on the knowledge that the modular function maps the upper half-plane locally biholomorphically onto $\mathbb{C} \setminus \{0, 1\}$.

Suppose that in the proof you replace the modular function by the exponential function and the set $\mathbb{C} \setminus \{0, 1\}$ by the set $\mathbb{C} \setminus \{0\}$. (The exponential function maps the upper half-plane locally biholomorphically onto $\mathbb{C} \setminus \{0\}$.) The modified proof seemingly implies the false conclusion that a non-constant entire function cannot omit *one* value from its range.

So which step in the proof breaks down in the modified situation?