Exercise on overconvergence

If a power series has radius of convergence equal to 1, then the sequence of partial sums diverges at every point z of modulus greater than 1. The power series is called *overconvergent* if some *subsequence* of partial sums converges at some point z with modulus greater than 1.

This exercise applies Runge's theorem to demonstrate the existence of a "universal" power series that overconverges to every holomorphic function.[1] More precisely, there exists a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence 1 and with the following remarkable property. For every closed disc Δ that is disjoint from the closed unit disc $\overline{D}(0,1)$, and for every function f that is holomorphic in an open neighborhood of Δ, there exists an increasing sequence of positive integers $\{n(k)\}_{k=1}^{\infty}$ such that the partial sums $\sum_{j=0}^{n(k)} c_j z^j$ converge uniformly on Δ to $f(z)$ as $k \to \infty$.

1. Use countability arguments to show the following.

(a) Let $\mathcal{P}$ be the set of holomorphic polynomials whose coefficients are complex numbers with rational real part and rational imaginary part. Then $\mathcal{P}$ is a countable set.

(b) There exists a countable set $\mathcal{B}$ of closed discs, each disjoint from the closed unit disc $\overline{D}(0,1)$, such that every closed disc that is disjoint from the closed unit disc $\overline{D}(0,1)$ has an open neighborhood that is contained in infinitely many of the discs in $\mathcal{B}$.

(c) The Cartesian product $\mathcal{B} \times \mathcal{P}$ can be enumerated as a sequence $\{(B_j, p_j)\}_{j=1}^{\infty}$, where each $B_j \in \mathcal{B}$ and each $p_j \in \mathcal{P}$.

2. Use Runge's theorem to construct inductively a sequence of polynomials $\{q_j\}_{j=1}^{\infty}$ such that every coefficient of every q_j has modulus less than 1, the lowest power of z in $q_j(z)$ is larger than the highest power of z in $q_{j-1}(z)$, and $\max\left\{\left|p_j(z) - \sum_{m=1}^{j} q_m(z)\right| : z \in B_j\right\} < 1/2^j$.

3. Show that the series $\sum_{j=1}^{\infty} q_j(z)$, viewed as a power series, has the required properties.

[1] Such examples were constructed independently by Wolfgang Luh [Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten, *Mitteilungen aus dem Mathematischen Seminar Giessen* **88** (1970)] and by Charles K. Chui and Milton N. Parnes [Approximation by overconvergence of a power series, *Journal of Mathematical Analysis and Applications* **36** (1971) 693–696].