Theory of Functions of a Complex Variable II

Part A

State *three* of the following theorems. When a theorem has multiple versions, you may state any one correct version.

(i) The Schwarz reflection principle.

(ii) Hadamard's gap theorem.

(iii) Montel's theorem about normal families (we saw two such theorems; pick either one).

(iv) Hurwitz's theorem about limits of sequences of holomorphic functions.

(v) Runge's approximation theorem.

Part B

Pick *three* of the following items. For each item, either construct an example satisfying the stated conditions or prove that no example exists (whichever is appropriate).

1. A function that is continuous on the closed unit disk and holomorphic on the open unit disk but cannot be continued analytically to a neighborhood of any boundary point of the disk.
2. A sequence $\{f_n\}_{n=1}^{\infty}$ of holomorphic functions on the open unit disk normalized such that $f_n(0) = 0$ and $f_n'(0) = 1$ for every n, yet there is no subsequence of these functions that converges uniformly on compact subsets of the disk.
3. A sequence of entire functions that converges uniformly on every compact subset of the real axis but does *not* have the property that the sequence converges uniformly on every compact subset of the complex plane.
4. A nonconstant entire function such that $f^{(n)}(n) = 0$ for every natural number n. (The exponent in parentheses indicates a derivative of high order.)
5. A surjective (but not injective) holomorphic map from an annulus to a disk.