

In this assignment, you will analyze an example of a holomorphic function in the unit disk that has the unit circle as natural boundary, yet all derivatives of the function extend to be continuous on the closed disk. The example reflects the significant difference between (real) functions that are class C^∞ (infinitely differentiable) and functions that are class C^ω (real analytic).

The example is the power series

$$\sum_{n=1}^{\infty} \frac{z^{2^n}}{n!} \quad (\text{notice exponent } 2^n, \text{ not } 2n). \quad (*)$$

This example is mentioned in the textbook on page 41 in Chapter 4, but without much explanation. Your task is to fill in the details, as follows.

1. The main technical difficulty in analyzing the series (*) is that $n!$ grows in a somewhat complicated way. Stirling's formula says that $n!$ is asymptotic to $(n/e)^n \sqrt{2\pi n}$, but you do not need such precise information here. A simpler, cruder estimate will do: namely,

$$(n/e)^n < n! \leq n^n.$$

The upper estimate is evident, for $n!$ is the product of n factors each no larger than n . Prove the lower estimate by taking logarithms, recognizing $\log(n) + \log(n-1) + \cdots + \log(2)$ as a Riemann sum, and comparing with $\int_1^n \log(x) dx$, an integral that you can compute exactly by integrating by parts.

2. Show that the series (*) has radius of convergence equal to 1.
3. Show that the series (*) converges uniformly on $\{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disk.
4. Show, more generally, that for every natural number k , the k th derivative of the series (*) converges uniformly on the closed unit disk.
5. Deduce from Hadamard's gap theorem that there is no point on the unit circle across which the series (*) can be continued analytically.
6. Can you prove the noncontinuability of (*) without using Hadamard's gap theorem?