In this assignment, you will apply ideas about analytic continuation to solve the following problem from a past qualifying examination:

> Show that there exists a holomorphic function f on the region $\mathbb{C} \setminus [-1, 1]$ such that $(f(z))^2 = z^2 - 1$ for all z in the region.

You may assume known the proposition that a zero-free holomorphic function in a simply connected region admits a holomorphic square root (which was proved in Math 617 by hand but alternatively is a consequence of the monodromy theorem). The difficulty in the problem is that this proposition does not directly apply, for the region $\mathbb{C} \setminus [-1, 1]$ is not simply connected.

Method A

1. Observe that the function $z^2 - 1$ admits a holomorphic square root in the open upper half-plane.
2. Apply the Schwarz reflection principle to extend a branch of $\sqrt{z^2 - 1}$ across the portion $(1, \infty)$ of the real axis; similarly extend $\sqrt{z^2 - 1}$ across the portion $(-\infty, -1)$ of the real axis.
3. Show that the two extensions agree in the lower half-plane, thus giving a globally defined function $\sqrt{z^2 - 1}$ on $\mathbb{C} \setminus [-1, 1]$.

Method B

1. Observe that the inversion $z \mapsto 1/z$ maps $\widehat{\mathbb{C}} \setminus [-1, 1]$ to the simply connected planar region $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, where $\widehat{\mathbb{C}}$ denotes $\mathbb{C} \cup \{\infty\}$, the extended complex numbers.
2. Argue that the simply connected region $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$ is the domain of a holomorphic function g such that $(g(z))^2 = 1 - z^2$ for all z in the region.
3. On the original region $\mathbb{C} \setminus [-1, 1]$, set $f(z)$ equal to $z g(1/z)$, and check that this f solves the problem.