Exercise on combining the Weierstrass and Mittag-Leffler theorems

Although the proof of the Weierstrass theorem (about prescribed zeroes) is multiplicative, and the proof of the Mittag-Leffler theorem (about prescribed poles and principal parts) is additive, the two theorems can be combined into a unified statement, as Mittag-Leffler himself showed.[1]

Suppose G is an open set in $\mathbb{C}$, and $(a_n)_{n=1}^{\infty}$ is a discrete sequence of distinct points in G. (The word "discrete" here means that the sequence has no accumulation point inside G.) Let $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ be two sequences of polynomials. The claim is that there exists a meromorphic function f on G having no zeroes or poles outside of the sequence (a_n) and such that for each n, the difference

$$f(z) - p_n(1/(z - a_n)) - q_n(z - a_n)$$

has (a removable singularity and) a zero at a_n of order at least $1 + \deg q_n$. In other words, you can prescribe not only the principal part of the Laurent series (the terms having negative exponents) but also finitely many terms having positive exponents. This statement generalizes Theorem 13.5 (holomorphic interpolation) from the textbook (which addresses the case that the polynomials p_n are identically zero).

Your task is to prove this statement by applying the versions that you already know of the Weierstrass theorem and the Mittag-Leffler theorem. Here is a suggestion for how the argument could go.

As a first step, use the Weierstrass theorem to find a holomorphic function g having for each n a zero at a_n of order $1 + \deg q_n$. Then use the Mittag-Leffler theorem to find a meromorphic function h having for each n the same principal part at a_n as the quotient

$$\frac{p_n(1/(z - a_n)) + q_n(z - a_n)}{g(z)}.$$

The first try is to set f_1 equal to the product gh. That procedure yields a proof of Theorem 13.5 (and more), bypassing the linear-algebra calculation in the textbook.

But you are not yet done, for the standard version of the Mittag-Leffler theorem does not control the location of zeroes. The function f_1 just determined might have some extraneous zeroes lying outside the sequence (a_n). Some additional trickery is needed to circumvent this difficulty.

Next apply the Weierstrass theorem to create a meromorphic function f_2 having zeroes and poles of the same orders as f_1 at the points of the sequence (a_n) and having no other zeroes or poles. (The Weierstrass theorem does not control the coefficients of the Laurent series of f_2, however.) Locally near each point a_n, you can define a holomorphic branch of $\log(f_1/f_2)$, and by the previous step there is a holomorphic function φ on G that agrees to suitably high order (how high?) at each a_n with $\log(f_1/f_2)$. Set f equal to $f_2 e^{\varphi}$.

[1] G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," *Acta Mathematica* **4** (1884) 1–79.