Exercise on three-circles theorems

In this assignment, you will apply subharmonic functions to prove Hadamard's three-circles theorem (which is one of the topics on the official syllabus for the qualifying examination in complex analysis).

1. Suppose u is a subharmonic function in an annulus with inner radius a and outer radius b: namely, $\{ z \in \mathbb{C} : a < |z| < b \}$. Let $m(r)$ denote the maximum of $u(z)$ when $|z| = r$. Prove that if $a < r_1 < r < r_2 < b$, then

   $$m(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} m(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} m(r_2).$$

   Sometimes this inequality is described by saying that $m(r)$ is "a convex function of $\log r$." An equivalent formulation of the inequality is that

   $$m(r) \leq \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}} m(r_1) + \frac{\log \frac{r}{r_1}}{\log \frac{r_2}{r_1}} m(r_2).$$

   Suggestion: Use a harmonic comparison function of the form $A + B \log |z|$.

2. Deduce that if f is a holomorphic function in the annulus, and $M(r)$ denotes the maximum of $|f(z)|$ when $|z| = r$, then

   $$M(r) \leq M(r_1)^{\lambda} M(r_2)^{1-\lambda}, \quad \text{where } a < r_1 < r < r_2 < b, \text{ and } \lambda = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}.$$

   This statement is *Hadamard's three-circles theorem* (Theorem 23.0.3 on page 387 of the textbook).

3. Suppose in Part 1 that u is bounded above, and $a = 0$. Let r_1 approach 0 to deduce that $m(r)$ is an increasing function of r. Extend u to the whole disk $\{ z \in \mathbb{C} : |z| < b \}$ by declaring $u(0)$ to be $\lim_{r \to 0^+} m(r)$. Prove that the extended function u is subharmonic on the whole disk.

   This statement is a *removable singularity theorem* for subharmonic functions that are bounded above on a punctured disk.