Midterm Examination

Instructions Solve *four* of the following six problems.

1. State
   a) the Riemann mapping theorem, and
   b) the Weierstrass factorization theorem.

2. For which values of the complex variable z does the infinite product $\prod_{n=1}^{\infty}(1 - z^n)$ converge? Explain.

3. Suppose f is an analytic function that maps $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$ (the right-hand half-plane) into $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk), and $f(1) = 0$. How big can $|f(2)|$ be?

4. Let $\mathcal{S}$ denote the set of injective analytic functions on the unit disk satisfying the following normalization: if $f \in \mathcal{S}$, then $f(0) = 0$ and $f'(0) = 1$. Results of A. Hurwitz and P. Koebe from over a century ago imply that $\mathcal{S}$ is a normal family.
   Assuming this theorem, deduce that the set $\left\{ \dfrac{1}{f'} : f \in \mathcal{S} \right\}$ of reciprocals of derivatives of functions in $\mathcal{S}$ is a normal family too.

5. Suppose $f(z) = \prod_{n=2}^{\infty} \left(1 - \dfrac{z}{n!}\right)$. Prove that $\lim_{|z|\to\infty} \dfrac{|f(z)|}{\exp(|z|)} = 0$.

6. Prove that a doubly connected open region bounded on the outside by a circle and on the inside by a square can be mapped biholomorphically to a region bounded by two smooth curves (that is, curves having at each point a well-defined tangent line).
   See the figure below.