Final Examination

Part A

State *three* of the following theorems.

1. Montel's theorem about locally bounded families of holomorphic functions.
2. Runge's theorem about polynomial approximation.
3. Hadamard's factorization theorem for entire functions.
4. Harnack's principle.
5. Picard's great theorem.

Part B

Pick *three* of the following items. For each item, either construct an example satisfying the stated conditions or prove that no example exists (whichever is appropriate).

1. A harmonic function u in the unit disk such that $\lim_{z \to e^{i\theta}} u(z) = 0$ whenever $0 < \theta < 2\pi$, but $\lim_{z\to 1} u(z)$ either fails to exist or is different from 0.
2. A holomorphic function f in the unit disk such that the image of f is the twice-punctured plane $\mathbb{C} \setminus \{0, 1\}$.
3. An entire function of finite genus with a zero of order $n!$ at each natural number n.
4. A continuous function that is subharmonic on the entire plane, constant on some nonvoid open subset, but not identically constant.
5. A sequence $\{p_n(z)\}_{n=1}^{\infty}$ of polynomials that is locally bounded in $\mathbb{C}$, and for each natural number n the degree of p_n equals n, and $\min\{\, |p_n(z)| : |z| \le n \,\}$ exceeds 1 for each n.