Exercise on the order of an entire function

The goal of this exercise is to deepen your understanding of the concept of the order of an entire function by comparing four definitions.

The notation $M_f(r)$, or $M(r)$ for short, means $\max\{\, |f(z)| : |z| \le r \,\}$. Liouville's theorem implies that if f is a nonconstant entire function, then $M(r) \to \infty$ as $r \to \infty$. The order of f characterizes how fast $M(r)$ goes to ∞.

Consider the following four numbers associated to a nonconstant entire function f whose Maclaurin series is $\sum_{n=0}^{\infty} c_n z^n$.

$$\rho := \inf \left\{\, t \in \mathbb{R} : |f(z)| \exp(-|z|^t) \text{ is a bounded function of } z \text{ in } \mathbb{C} \,\right\}$$

$$\sigma := \inf \left\{\, t \in \mathbb{R} : \lim_{r\to\infty} \frac{\log M(r)}{r^t} = 0 \,\right\}$$

$$\lambda := \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

$$\beta := \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|c_n|}} \qquad \text{(When } |c_n| = 0\text{, interpret the whole fraction as being 0.)}$$

The main goal is to show that $\rho = \sigma = \lambda = \beta$. (The choice of letters is ad hoc. There is no entirely standard notation for these four quantities.)

1. To check that you understand the definitions, verify that when $f(z) = ze^z$, each of the four numbers ρ, σ, λ, and β is equal to 1.

2. Let s be an arbitrary positive real number, and suppose that $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/s}}$. Show that the value of β for this function f is equal to s. (Thus, the knowledge that β is equal to ρ reveals that an entire function of arbitrary positive order exists.)

Now let f be a general nonconstant entire function, and fix a positive ε.

3. Show that if ρ is finite, then $\log M(r)$ is bounded above by a constant plus $r^{\rho+\varepsilon}$. Deduce that $\sigma \le \rho + 2\varepsilon$.

4. Show that if σ is finite, then $\log M(r) < r^{\sigma+\varepsilon}$ for sufficiently large r. Deduce that $\lambda \le \sigma + \varepsilon$.

5. Show that if λ is finite, then $M(r) < \exp(r^{\lambda+\varepsilon})$ for sufficiently large r. Bound $|c_n|$ for large n by applying Cauchy's estimate with $r = n^{1/(\lambda+\varepsilon)}$. Deduce that $\beta \le \lambda + \varepsilon$.

6. Show that if β is finite, then $|c_n| < n^{-n/(\beta+\varepsilon)}$ for sufficiently large n. Observe that $M(r) \le \sum_{n=0}^{\infty} |c_n|\, r^n$. By splitting the sum where $n \approx (2r)^{\beta+\varepsilon}$, show that $\rho \le \beta + 2\varepsilon$.

Finally, let $\varepsilon \downarrow 0$ to deduce that $\rho = \sigma = \lambda = \beta$.