# Math 650-600: Several Complex Variables

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## Exercise on Reinhardt domains

For a complete Reinhardt domain in $\mathbb{C}^2$ with smooth boundary, show that the Levi form is $\geq 0$ if and only if the domain is logarithmically convex.

This solves a special case of the Levi problem.



## Exercise on tube domains

An unbounded domain $\Omega$ in $\mathbb{C}^n$ is called a tube domain with base $G$ in $\mathbb{R}^n$ if $\Omega = \{\, x + iy \in \mathbb{C}^n : x \in G \text{ and } y \in \mathbb{R}^n \,\}$.

**Exercise.** Show that a tube domain in $\mathbb{C}^n$ is pseudoconvex if and only if the base $G$ in $\mathbb{R}^n$ is convex.

## Recap from last time

**Theorem.** Suppose $\Omega$ is a pseudoconvex domain in $\mathbb{C}^n$. Then for every $\bar\partial$-closed $(0,1)$-form $f$ with coefficients of class $C^\infty(\Omega)$, there exists a function $u$ of class $C^\infty(\Omega)$ such that $\bar\partial u = f$.

**Extension theorem.** Suppose $\Omega$ is a pseudoconvex domain in $\mathbb{C}^n$, and let $\omega$ be the intersection of $\Omega$ with a complex hyperplane. For every holomorphic function $f$ on $\omega$ (in $\mathbb{C}^{n-1}$), there is a holomorphic function $F$ on $\Omega$ such that $F|_\omega = f$.

Idea of proof: Extend $f$ smoothly. Then solve a $\bar\partial$-problem to adjust the extension, making it holomorphic.



## Solution of the Levi problem

**Theorem.** Every pseudoconvex domain in $\mathbb{C}^n$ is a domain of holomorphy.

**Proof by induction on the dimension $n$.** The basis step ($n = 1$) is easy: every domain in $\mathbb{C}^1$ is a domain of holomorphy. Suppose the result is know for dimensions less than $n$, and let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$.

It suffices to show that whenever an open ball contained in $\Omega$ has a boundary point $p$ lying in $b\Omega$, there is a holomorphic function on $\Omega$ that is unbounded on the radius terminating at $p$.

Fix such a ball and point $p$, and slice $\Omega$ with a complex hyperplane containing the radius terminating at $p$. The slice is pseudoconvex, so by the induction hypothesis, there is a holomorphic function $f$ on the slice that is unbounded on the radius. Extend $f$ to a holomorphic function $F$ on $\Omega$ by the slice extension theorem.