Math 650-600: Several Complex Variables

Harold P. Boas
boas@tamu.edu

Final class meeting

Our last meeting will be on the afternoon of the redefined day, Tuesday 3 May, 3:15–4:30 in Milner 313.

Recap from last time

Solving the $\bar{\partial}$-equation in $C^\infty(\Omega)$ for $(0,1)$-forms on a pseudoconvex domain Ω reduces to constructing weight functions φ_1, φ_2, and φ_3 for which one has the key estimate

$$\|f\|_{\varphi_2} \le \|\bar{\partial}^* f\|_{\varphi_1} + \|\bar{\partial} f\|_{\varphi_3}$$

when $f = \sum_{j=1}^n f_j \, d\bar{z}_j$ is in $\operatorname{Dom} \bar{\partial}^* \cap \operatorname{Dom} \bar{\partial}$.
By a clever choice of the weight functions, we reduce to proving the key estimate when f has compactly supported smooth coefficients, so we can integrate by parts without boundary terms.

We will construct two functions φ and ψ and set $\varphi_3 = \varphi$, $\varphi_2 = \varphi - \psi$, $\varphi_1 = \varphi - 2\psi$, which means that $\varphi_3 - \varphi_2 = \psi$ and $\varphi_2 - \varphi_1 = \psi$. The role of ψ is to guarantee density of $C_0^\infty(\Omega)$ in $\operatorname{Dom} \bar{\partial}^* \cap \operatorname{Dom} \bar{\partial}$, while φ exploits the pseudoconvexity of Ω.

The calculation in progress

Explicit computation showed that

$$\sum_{j,k}\big(\langle\delta_j f_j,\delta_k f_k\rangle_\varphi-\langle\tfrac{\partial f_j}{\partial\bar z_k},\tfrac{\partial f_k}{\partial\bar z_j}\rangle_\varphi\big)\le\|\bar\partial f\|^2_{\varphi_3}+2\|\bar\partial^* f\|^2_{\varphi_1}+2\int_\Omega|f|^2\,|\partial\psi|^2\,e^{-\varphi},$$

where $\delta_j f_j=e^{\varphi}\frac{\partial}{\partial z_j}(e^{-\varphi}f_j)$. Since we may assume that f has compact support, two integrations by parts convert the left-hand side to

$$\sum_{j,k}\big\langle\big(\delta_j\tfrac{\partial}{\partial\bar z_k}-\tfrac{\partial}{\partial\bar z_k}\delta_j\big)f_j,f_k\big\rangle_\varphi$$

and we want to show that this term equals

$$\sum_{j,k}\left\langle\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}f_j,f_k\right\rangle_\varphi.$$

Computation of the commutator term

For a smooth function u, the definition of the operator δ_j gives

$$\left(\delta_j\frac{\partial}{\partial\bar z_k}-\frac{\partial}{\partial\bar z_k}\delta_j\right)u=e^{\varphi}\frac{\partial}{\partial z_j}\left(e^{-\varphi}\frac{\partial}{\partial\bar z_k}u\right)-\frac{\partial}{\partial\bar z_k}\left(e^{\varphi}\frac{\partial}{\partial z_j}(e^{-\varphi}u)\right)$$

$$=e^{\varphi}\frac{\partial}{\partial z_j}\frac{\partial}{\partial\bar z_k}(e^{-\varphi}u)+e^{\varphi}\frac{\partial}{\partial z_j}\left(\frac{\partial\varphi}{\partial\bar z_k}e^{-\varphi}u\right)-\frac{\partial}{\partial\bar z_k}\left(e^{\varphi}\frac{\partial}{\partial z_j}(e^{-\varphi}u)\right)$$

$$=e^{\varphi}\frac{\partial}{\partial z_j}\frac{\partial}{\partial\bar z_k}(e^{-\varphi}u)+e^{\varphi}\frac{\partial}{\partial z_j}\left(\frac{\partial\varphi}{\partial\bar z_k}e^{-\varphi}u\right)$$

$$-e^{\varphi}\frac{\partial}{\partial\bar z_k}\frac{\partial}{\partial z_j}(e^{-\varphi}u)-e^{\varphi}\frac{\partial\varphi}{\partial\bar z_k}\frac{\partial}{\partial z_j}(e^{-\varphi}u).$$

The first and third terms cancel. The product rule implies that

$$e^{\varphi}\frac{\partial}{\partial z_j}\left(\frac{\partial\varphi}{\partial\bar z_k}e^{-\varphi}u\right)-e^{\varphi}\frac{\partial\varphi}{\partial\bar z_k}\frac{\partial}{\partial z_j}(e^{-\varphi}u)=\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}u.$$

The final inequality

Putting the pieces together, we have

$$\int_\Omega \sum_{j,k} \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} f_j \overline{f_k} - 2|f|^2 \, |\partial \psi|^2 \right) e^{-\varphi_3} \le \|\bar{\partial} f\|^2_{\varphi_3} + 2\|\bar{\partial}^* f\|^2_{\varphi_1}.$$

Choosing φ such that

$$\sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} f_j \overline{f_k} \ge 2|f|^2 (|\partial \psi|^2 + e^{\psi})$$

makes the left-hand side $\ge 2\|f\|^2_{\varphi_2}$ since $\psi = \varphi_3 - \varphi_2$.

Thus we have the key estimate, and hence the solvability of the $\bar{\partial}$-equation, if we can construct suitable functions ψ and φ.

Construction of φ

Supposing the function ψ to be fixed, we need to construct a very strongly plurisubharmonic function φ such that

$$\sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k \ge 2|w|^2 (|\partial \psi|^2 + e^{\psi}) \qquad \text{for all } w \text{ in } \mathbb{C}^n.$$

Suppose we already have a C^∞ plurisubharmonic exhaustion function φ_0 for Ω. By adding $|z|^2$ to φ_0, we can ensure that $\sum_{j,k} \frac{\partial^2 \varphi_0}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k \ge |w|^2$ for all w in $\mathbb{C}^n$.

If χ is a convex C^∞ function of one real variable, then

$$\sum_{j,k} \frac{\partial^2 (\chi \circ \varphi_0)}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k \ge (\chi' \circ \varphi_0) \sum_{j,k} \frac{\partial^2 \varphi_0}{\partial z_j \partial \bar{z}_k} w_j \overline{w}_k \ge (\chi' \circ \varphi_0)\, |w|^2.$$

If χ is sufficiently rapidly increasing, then $\varphi := \chi \circ \varphi_0$ works.

Smooth plurisubharmonic exhaustion

Let u be a continuous plurisubharmonic exhaustion function, say $u(z) = \max(-\log \operatorname{dist}(z, b\Omega), |z|^2)$. By convolving with a mollifier and adding $\epsilon|z|^2$, we can get a function u_j in $C^\infty(\Omega)$ such that on $\{ z \in \Omega : u(z) \le j+1 \}$ the function u_j is strictly plurisubharmonic and $u(z) < u_j(z) < u(z) + 1$.

Compose u_j with a convex, C^∞, (weakly) increasing function χ_j of one real variable such that $\chi_j(t) = 0$ when $t < j$ and χ_j is rapidly increasing when $t > j$. The sum $\sum_{j=1}^{\infty}(\chi_j \circ u_j)$ is locally finite, and so defines a C^∞ function that, for suitable choices of the χ_j, is an exhaustion function that is plurisubharmonic on each set $\{ z \in \Omega : j \le u(z) \le (j+1) \}$.

Thus every pseudoconvex domain in $\mathbb{C}^n$ admits a C^∞ strictly plurisubharmonic exhaustion function.