Announcement

Math Club Meeting

Complex Numbers and Geometry

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Thursday, February 17, 6:30 PM

Blocker 627

Solution of the $\overline{\partial}$-problem in the plane

Reminder from last time

If g has continuous first partial derivatives on the closure of a bounded domain G in $\mathbb{C}$, then a solution of the equation $\frac{\partial f}{\partial \bar{z}} = g$ is given by

$$f(z) = \frac{1}{2\pi i} \int_G \frac{g(\zeta)}{\zeta - z} \, d\zeta \wedge d\overline{\zeta}.$$

The proof follows from Cauchy's integral formula with remainder.

Exercises on $\overline{\partial}$ in the plane

1. Find an explicit (smooth) solution to the equation $\partial f/\partial\bar{z} = 1/z$ in the punctured plane $\mathbb{C} \setminus \{0\}$.
2. Find an explicit (smooth) solution to the equation $\partial f/\partial\bar{z} = 1/\bar{z}$ in the punctured plane $\mathbb{C} \setminus \{0\}$.

Natural boundary in the plane

Example. The power series $\sum_{n=1}^{\infty} \frac{1}{n!} z^{2^n}$ represents a holomorphic function in the open unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ that extends to be a class C^∞ function on the closed unit disc $\{ z \in \mathbb{C} : |z| \leq 1 \}$ but that does not admit an analytic continuation to a neighborhood of any boundary point.

The fact that the unit circle is the natural boundary for this series follows (for instance) from the *Hadamard gap theorem*:
If $\{p_n\}$ is an exponentially increasing sequence of positive integers, then the series $\sum_{n=1}^{\infty} a_n z^{p_n}$ does not extend holomorphically across any boundary point of its disc of convergence.

Here "exponentially increasing" means that there exists a positive number λ such that $p_{n+1} \geq (1 + \lambda) p_n$ for all n.

Domains of holomorphy

Theorem/definition. A connected open set Ω in $\mathbb{C}^n$ is called a domain of holomorphy if either of the following two equivalent properties holds.

1. There exists a holomorphic function on Ω that does not extend holomorphically across any part of the boundary.
2. For every point p in the boundary of Ω, there exists a holomorphic function on Ω that does not extend holomorphically across the boundary at p.

Obviously (1) $\implies$ (2). We will prove the converse later.

The precise meaning of "f does not extend holomorphically across the boundary at p" is "there do not exist a connected neighborhood U of p and a non-empty open subset V of $U \cap \Omega$ and a holomorphic function F on U such that $F|_V = f$."

Solving $\overline{\partial}$ when $n \geq 2$

Theorem. When $n \geq 2$, if $g = \sum_{j=1}^{n} g_j(z)\, d\overline{z}_j$ satisfies the compatibility condition $\overline{\partial} g = 0$, and if each g_j is a continuously differentiable function having compact support in $\mathbb{C}^n$, then there is a compactly supported continuously differentiable function f such that $\overline{\partial} f = g$.

Proof. Set $f(z_1, \ldots, z_n) = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\overline{\zeta}$.
Then $\partial f / \partial \overline{z}_1 = g_1$ by the one-variable case previously considered. When $j > 1$, pass the derivative $\partial / \partial \overline{z}_j$ under the integral sign and use the compatibility condition to replace $\partial g_1 / \partial \overline{z}_j$ with $\partial g_j / \partial \overline{\zeta}$. The one-variable Cauchy formula with remainder implies that $\partial f / \partial \overline{z}_j = g_j$.
Finally, f has compact support because f is holomorphic outside a compact set, and f vanishes when $|z_n|$ is large.

Exercise: $1 \neq 2$

In contrast to the situation when $n \geq 2$, show that in $\mathbb{C}$, if g has compact support but $\int_{\mathbb{C}} g(z)\, dz \wedge d\bar{z} \neq 0$, then no function f such that $\partial f/\partial \bar{z} = g$ can have compact support.